The Delta-M Method: Rapid Yet Accurate Radiative Flux Calculations for Strongly Asymmetric Phase Functions

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ABSTRACT

The delta-M method represents a natural extension of the recently proposed delta-Eddington approximation to all orders M of angular approximation. It relies essentially on matching the first $2M$ phase function moments and using a Dirac delta-function representation of forward scattering. Computed fluxes are remarkably accurate at very low orders M of approximation, even when the phase function is strongly asymmetric; thus the associated $M \times M$ matrix computations remain small and manageable. Flux is automatically conserved, making phase function "renormalization" unnecessary. Phase function truncation is effected in a much more attractive manner than in the past; furthermore, truncation tends to zero as $M \to \infty$. Errors are shown to oscillate with (roughly) exponentially decreasing amplitude as M increases; which has the curious consequence that increasing M by small amounts does not necessarily reduce error. Mie computations associated with the δ-M method can be considerably reduced, based on a simple technique for phase function moment calculations proposed herein.

1. Angular error in multiple-scattering calculations

The difficulties of performing radiative transfer calculations for strongly asymmetric phase functions are too well known to need belaboring [see, e.g., the review articles of Hunt (1971b), Hansen and Travis (1974) and Irvine (1975)]. A direct attack on the problem can consume enormous amounts of computer time and storage space, as exemplified by the work of Dave and Canosa (1974) and Hansen (1969b) among others.

Furthermore, the error properties of current computational methods are not well understood. Irvine (1975) remarks that "it is difficult to obtain an *a priori* idea of the accuracy of a given order of approximation." In a similar vein, Hunt (1971b) remarks that, "In the past, many authors . . . have provided the reader with a somewhat vague measure of the accuracy of the results produced, or even allowed him to infer for himself the accuracy of the procedure. . . ."

In order to remedy these problems of excessive computation and vague accuracy, this paper presents a new technique, the "δ-M method," for efficiently computing fluxes in scattering media with strongly asymmetric phase functions. The δ-M method follows logically from the recently proposed δ-Eddington approximation (Joseph *et al.*, 1976; Wiscombe and Joseph, 1977).

Simply stated, the computational difficulties posed by strongly asymmetric phase functions are all a consequence of the fact that such phase functions cannot be represented by polynomials of low degree. As a result, intervening quadrature rules, series expansions, systems of differential and algebraic equations, matrix operations, etc., all become large and unwieldly.

Consider the plane-parallel, monochromatic radiative transfer equation

$$\mu \frac{\partial I}{\partial \tau} = -I + \omega J, \tag{1}$$

where the scattering term is

$$J \equiv \int_{\Omega} P(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') I(\tau, \boldsymbol{\Omega}') \frac{d\Omega'}{4\pi} \tag{2}$$

and where the following notation is used:

$I(\tau,\boldsymbol{\Omega})$	radiation intensity
τ	optical depth
$\boldsymbol{\Omega}$	unit direction vector
μ	cosine of zenith angle of $\boldsymbol{\Omega}$ (with respect to vertical)
ϕ	azimuth angle of $\boldsymbol{\Omega}$ (around the vertical)
ω	albedo for single scattering
$P(\cos\theta)$	scattering phase function (depending implicitly on τ)
θ	scattering angle.

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Consider $P(\cos\theta)$ to be expanded in (or at least adequately approximated by) an N-term series of Legendre polynomials P_n:

$$P(\cos\theta)=\sum_{n=0}^{N}(2n+1)\chi_n P_n(\cos\theta), \tag{3}$$

where the coefficients χ_n are the moments of P with respect to the Legendre polynomials

$$\chi_n=\frac{1}{2}\int_0^{\pi}P(\cos\theta)P_n(\cos\theta)\sin\theta d\theta. \tag{4}$$

Then the standard procedure (Chandrasekhar, 1960, p. 150) is to expand the intensity in an N-term Fourier cosine series in azimuth, i.e.,

$$I(\tau,\Omega)=\sum_{m=0}^{N}I_m(\tau,\mu)\cos m\phi. \tag{5}$$

[This is exact, given Eq. (3); I_{N+1}, I_{N+2}, etc., vanish identically.]

Thus the azimuthal series (5) has N terms when the phase function (3) is of degree N. This is obviously burdensome when N is large. Only the I_0 term contributes to fluxes; it satisfies the equation

$$\mu\frac{\partial I_0}{\partial\tau}=-I_0+\omega J_0, \tag{6}$$

where

$$J_0\equiv\frac{1}{2}\int_{-1}^{1}\bar{P}(\mu,\mu')I_0(\tau,\mu')d\mu', \tag{7}$$

and where $\bar{P}$ is the azimuthally averaged phase function

$$\bar{P}(\mu,\mu')\equiv\frac{1}{2\pi}\int_0^{2\pi}P[\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi]d\phi \tag{8a}$$

$$=\sum_{n=0}^{N}(2n+1)\chi_n P_n(\mu)P_n(\mu'). \tag{8b}$$

A variety of computational techniques are used to solve Eq. (1) (cf. Irvine, 1975). Of these, perhaps the most versatile and general (from the standpoint of being applicable to *all* values of optical depth and single-scattering albedo) are discrete ordinates, its close cousin spherical harmonics and adding-doubling. These three approaches have been developed vigorously by Liou (1973), Dave and Canosa (1974) and Grant and Hunt (1968), respectively. While diverging as to their details, these methods share a common starting point—namely, the integral J_0 [Eq. (7)] is approximated by a sum, assuming that the integrand is close to a polynomial. There is nothing wrong in principle with such an approach; Weierstrass's theorem assures us that every continuous function is arbitrarily closely approximable by a polynomial of sufficiently high degree. The only question is: is "sufficiently high" so high that the whole calculation becomes ponderous or even impossible?

We now define

n_0 = degree of "sufficiently accurate" polynomial approximation to integrand $\bar{P}I_0$ of J_0,
n_1 = number of terms in sum replacing J_0.

Clearly $\bar{P}$ is an Nth degree polynomial in μ' [Eq. (8b)], so its product with I_0 is at least an Nth degree polynomial in μ' ($n_0>N$). Also, in general, n_1 is on the order of n_0 (e.g., for Gaussian quadrature of J_0, $n_1\approx\frac{1}{2}n_0$). Summing up, we have

$$n_1\sim n_0>N.$$

But in spite of this heuristic argument, most investigators take $n_1\ll N$. The reason is threefold: first, computer time requirements escalate roughly as n_1^3 (Hansen, 1971) and storage requirements as n_1^2, although these vary somewhat from method to method. Second, numerical ill-conditioning often poses an upper limit to n_1, which is on the order of 20–30 for the discrete ordinate method (Liou, 1973; Irvine, 1975), and which necessitates lengthy "stabilizing transformations" in the case of the spherical harmonics method (Dave and Canosa, 1974). Third, round-off error growth is often such that it would be dangerous to take $n_1\sim N$ even if the first two objections were eliminated. For these reasons, almost all authors, even those with extensive computing resources, keep $n_1\lesssim 100$.

It is perhaps instructive to note values of n_1 used by some well-known investigators in this field. van de Hulst and Grossman (1968) used $n_1=14$–28 for $g=0.75$, and $n_1=48$–64 for $g=0.875$ Henyey-Greenstein phase functions. Hansen (1969b) used $n_1=40$ for an $N=50$-term aerosol phase function, and $n_1=96$ for an $N=140$-term cloud phase function; he generally increases n_1 until the results "suggest" errors are $\lesssim 0.1\%$ (Hansen, 1969a). Liou (1973) takes $n_1=4$–16; Twomey *et al.* (1966), $n_1=20$; and Grant and Hunt (1968), typically $n_1=14$–16. (The small values of n_1 in the last group of papers are probably based on the empirical observation that increasing n_1 beyond 20 or so does not obviously reduce the error.)

All investigators who take $n_1\ll N$ hope that they can work to within a few percent accuracy, either by employing artifices which effectively reduce N, or by increasing n_1 in steps and watching how the results change. Hunt (1971b) gives quite a candid summary of the latter procedure: ". . . it is not possible to give a precise value of $m[=\frac{1}{2}n_1]$ to be used . . . This can only be determined by repeating the experiment for various values of m, and carefully assessing the changes in the computed results." But this means stepping n_1 through a series of values all

of which are $\ll N$. Thus it is not surprising that a normal sort of convergence, in which the level of accuracy increases with n_1, is not observed. Hunt (1971a) provides dramatic illustrations of this: in stepping n_1 from 16 to 64, for phase functions with $N=223$ and 341, he finds no evidence of convergence beyond one significant figure even at $n_1=64$. Dave and Canosa (1974) show flux values which seem to have converged to two significant digits at $n_1=10$–20 but which continue to oscillate in the third or fourth significant digit from there all the way up to $n_1=100$. Potter (1970), using doubling for an $N=350$-term phase function, found absorptivities up to 0.6% for a nonabsorbing layer when he took $n_1=96$; therefore his fluxes are converged to only two significant digits even with such a large n_1.

The primary technique for reducing N is "truncation" of the forward peak from the phase function (reviewed by Hunt, 1971b). A second procedure is phase function "renormalization" (reviewed by Wiscombe, 1976). Both procedures are *ad hoc*, and none of the proposed variants (there are at least four for renormalization alone) is demonstrably superior to any other. Indeed, the number of reasonable truncation and renormalization procedures is limited only by one's imagination.

Hansen (1969a) and Potter (1970) truncate by extrapolating the "shoulder" of the Mie phase function into the forward peak region. Two other methods are given in this paper (Sections 3 and 4) and Weinman (1968) gives yet a fourth. Potter's truncation reduces N from 350 to 50, enabling him to take $n_1=24$ instead of $n_1=96$. Both Hansen and Potter quote flux errors (caused by truncation) in the range 0.3–1%; errors tend to be worst for small optical depth and/or large solar zenith angles.

Renormalization performs (hopefully small) corrections to the azimuthally averaged phase function $\bar{P}$ [Eq. (8)] in order to conserve flux. That is, it is designed to ensure

$$\frac{dF}{d\tau}=\frac{d}{d\tau}\left[2\pi\int_{-1}^{1}\mu I_0 d\mu\right]=0, \tag{9}$$

when $\omega=1$, where F is net flux. From the transfer equation (6), this means that

$$\int_{-1}^{1}(J_0-I_0)d\mu=0. \tag{10}$$

Eq. (10) is identically satisfied by the *exact* form of J_0 when the phase function is normalized to unity:

$$\frac{1}{2}\int_{-1}^{1}\bar{P}(\mu,\mu')d\mu=1 \tag{11}$$

which, from Eq. 8(b), is equivalent to $\chi_0=1$. Renormalization forces Eq. (10) to be identically satisfied *after all integrals have been approximated by sums.* The large spurious absorptions in Potter's (1970) calculations indicates the price one pays for not renormalizing. Dave and Canosa (1974) exhibit spurious absorption in the third or fourth significant digit, for the same reason. Renormalization not only conserves energy but also allows the use of smaller values of n_1; Hansen (1971) notes a case where renormalization enabled him to reduce n_1 from 60 to 28 with no loss in accuracy.

Section 4 lays out the theoretical basis of the δ-M method, which performs truncation in a much more natural manner than heretofore, and which does not require a separate renormalization step. The shortcomings of previous methods are described in Section 3, and the error behavior of the δ-M method is given in Sections 5 and 6, using sample phase functions taken from Section 2.

2. Phase functions used in examples

We shall employ three phase functions, $P(\cos\theta)$, in assessing the performance of the $\delta-M$ method. They are shown plotted versus θ in Fig. 1a and versus $\cos\theta$ in Fig. 1b. Two of them are Henyey-Greenstein phase functions,

$$P_{\mathrm{HG}}(\cos\theta)=\sum_{n=0}^{\infty}(2n+1)g^n P_n(\cos\theta) \tag{12}$$

(Joseph *et al.*, 1976), with asymmetry factors $g=0.75$ and $g=0.85$. The third is a Mie phase function for wavelength 0.5 μm, index of refraction 1.335, a gamma size distribution with effective radius 10 μm and effective variance 0.2 (cf. Hansen and Travis, 1974), and an asymmetry factor 0.863. This Mie phase function is typical for a stratiform cloud in the visible spectrum.

The representation in Fig. 1a is the one customarily used, since it makes the forward peak more visible. But the representation in Fig. 1b is more truthful, in that all our theories use the phase function as a function of $\cos\theta$, not θ. Fig. 1b makes it much more visually apparent why truncation is useful; the forward peak, especially for the Mie phase function, looks very much like a spike, for which $\delta(1-\cos\theta)$ is a good approximation.

The Mie and Henyey-Greenstein phase functions were chosen because they are at somewhat opposite extremes in terms of structure: the Mie function has about as much structure as one is likely to encounter in a polydispersed cloud or aerosol; while by contrast the Henyey-Greenstein functions are very smooth, have considerably lower and broader forward peaks, and are shaped more like phase functions for smaller, moderate to highly absorbing particles. The Legendre polynomial expansions are consequently of quite different lengths: $N\approx 900$ for the Mie function, while $N\approx 86$ ($g=0.85$) and $N\approx 49$ ($g=0.75$) for the Henyey-

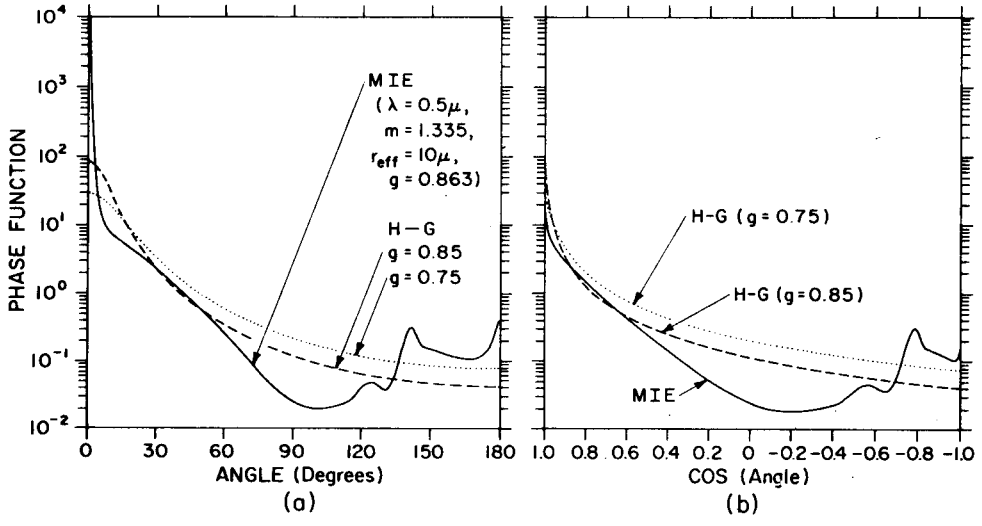

FIG. 1. The three phase functions used in various examples, plotted as functions of (a) θ and (b) $\cos\theta$. Two are Henyey-Greenstein $g=0.75$ and 0.85 cases, with values at $\theta=0$ of 28 and 86, respectively. The third is a Mie case for a polydisperse water cloud of effective radius 10 μm at wavelength 0.5 μm, with a value of 9.7×10^3 at $\theta=0$.

Greenstein cases. [These values of N are based on the somewhat arbitrary criterion that $\chi_n<10^{-6}$ for $n\gtrsim N$, cf. Eq. (4), and are rough values for illustrative purposes only.]

3. Critique of truncation and renormalization as previously practiced

The truncation and renormalization methods used in the past (see the Introduction) suffer from numerous deficiencies. First, they are completely *ad hoc*—they are only convenient patches, not permanent solutions. Second, there are several possible variants of each, which is confusing. Third, it is impossible to separate errors induced by truncation and renormalization from errors due to taking too small a value of n_1; the errors are all scrambled together. Fourth, both procedures stop convergence (as $n_1\to\infty$) from proceeding past, *at best*, the second or third significant digit. Fifth, truncation is independent of n_1; it should rather diminish as $n_1\to\infty$. Sixth, using less truncation, or a different truncation method, or a different renormalization method, may make errors smaller *or* larger in a way which has not at all been investigated. Finally, it has not been shown that renormalization corrections uniformly approach zero as $n_1\to\infty$, for any of the methods; indeed, for the Grant method (Wiscombe, 1976) we have observed a subset of these corrections growing without bound as $n_1\to\infty$. In sum, both truncation and renormalization, as presently practiced, seem intensely unsatisfactory.

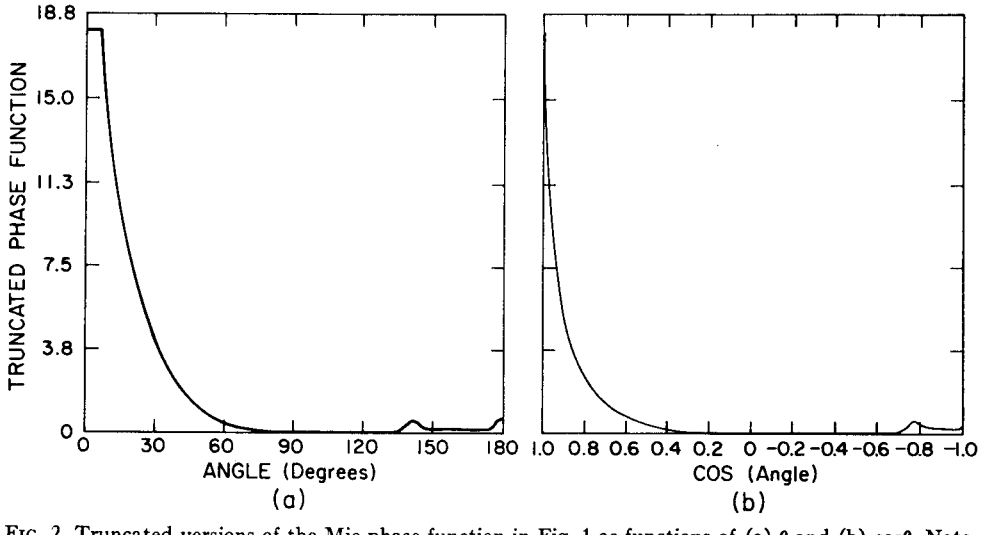

FIG. 2. Truncated versions of the Mie phase function in Fig. 1 as functions of (a) θ and (b) $\cos\theta$. Note that the vertical scale is linear rather than logarithmic here.

Perhaps an example may serve to make the above criticisms more specific. Let the Mie phase function in Fig. 1 be truncated as shown in Fig. 2, at an angle of 7.5°, causing 47% of the total of scattered photons to be presumed unscattered. The truncation shown consists merely in "lopping off" the forward peak with a horizontal line; the derivative discontinuity produced thereby is irrelevant, since nowhere in radiative transfer theory is the phase function ever differentiated. To maintain derivative continuity, as do Hansen (1969a) and Potter (1970), would require truncation to start at ~20° instead of 7.5°, thereby entailing a considerably greater mutilation of the phase function. Note again how much more sensible truncation appears when represented properly, as in Fig. 2b.

Consider a homogeneous nonabsorbing layer of optical depth τ, having the phase function of Fig. 2. Let calculations be performed for it with the diamond-initialized doubling method (Wiscombe, 1976). Benchmark results (for the full Mie phase function of Fig. 1) having at least four significant digits are obtained from the δ-M method of Section 4.

We find that the use of the fixed truncation of Fig. 2 virtually stops convergence as $n_1 \to \infty$. The effect is particularly severe for angles of incident radiation near grazing. For $\mu_0 = \cos(\text{solar zenith angle}) = 0.1$, albedo accuracy for $\tau = 0.1$ never improves beyond 2D (two decimal places), while n_1 increases from 8 to 120; for $\mu_0 = 0.1$ and $\tau = 1$–10, accuracy never improves past 3D while n_1 increases from 16 to 120. For $\mu_0 \leqslant 0.5$ and any optical depth, it is impossible to get more than 3D, or occasionally 4D, no matter how far one increases n_1. For $\tau = 0.1$, the $n_1 = 100$ solution ranges in accuracy from 2D for $\mu_0 = 0.1$ to 6D for $\mu_0 = 1$; but for $\tau = 1$–10, the range is reduced to 2–3D. Thus accuracy tends to deteriorate as optical depth and/or zenith angle increase, and it rises above 3D only for the very special case of optically thin layers at near-normal incidence.

Another experiment was made in which the Mie phase function was truncated at 3.8° rather than 7.5°. This *enhances* error in calculated albedos, compared to the 7.5° case, for $n_1 \leqslant 20$, and reduces error by a factor of 2–5 at $n_1 = 100$. Thus one does not know *a priori* if changing the amount of truncation will improve or worsen one's results.

The Wiscombe (1976) renormalization was used in the experiments above. It produces corrections almost an order of magnitude smaller than the Grant method. Unlike the Grant method, its corrections ϵ_i fall off uniformly, albeit slowly, as $n_1 \to \infty$. For the 7.5° truncation and for $n_1 = 8$, $|\epsilon_i| \leqslant 0.02$; for $n_1 = 16$, $|\epsilon_i| \leqslant 0.003$; and for $n_1 = 100$, $|\epsilon_i| \leqslant 0.0001$. For the 3.8° truncation, the ϵ_i increase almost an order of magnitude, on the average, even at $n_1 = 100$. Thus one is between Scylla and Charybdis—lessening the mutilation due to truncation only exaggerates that due to renormalization. Changing the renormalization method may also have a pronounced effect, though only for smaller values of n_1: changing from a Grant to a Wiscombe method changes computed albedos in anywhere from the second to the fifth decimal place, when $n_1 \leqslant 8$; but by $n_1 = 16$, these changes are already confined to the fourth through sixth decimal places. The changes are larger for layers of larger optical depth.

Our overall impression, garnered from these and many other numerical experiments, is that the conventional truncation-renormalization approach can be relied on for only 1–2 significant digits of accuracy, especially for the smaller values of n_1 ($\leqslant 20$, say) that most investigators would prefer to use. It *may* be more accurate in some situations, as we have indicated above, but one can not *rely* on that. Furthermore, it is entirely unsatisfactory for ascertaining the limit as $n_1 \to \infty$.

4. The δ-M method

In the following, we shall assume the reader is familiar with the delta-Eddington paper of Joseph *et al.* (1976) in order to avoid needless repetition.

a. Derivation

We approximate the phase function $P(\cos\theta)$ by[2]

$$P^*(\cos\theta) \equiv 2f\delta(1-\cos\theta) + (1-f)\sum_{n=0}^{2M-1}(2n+1)\chi_n^* \times P_n(\cos\theta), \quad (13)$$

where M is essentially the order of the approximation ($M=1$ leads to the delta-Eddington approximation) and where the coefficients are determined by matching the moments of P^* to those χ_m of P [cf. Eq. (4)]:

$$\chi_m = \frac{1}{2}\int_0^\pi P^*(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \begin{cases} f+(1-f)\chi_m^*, & m \leqslant 2M-1 \\ f, & m \geqslant 2M \end{cases}$$

which may be solved *uniquely* for the coefficients χ_m^*, i.e.,

$$\chi_m^* = \frac{\chi_m - f}{1-f}, \quad m = 0, \ldots, 2M-1 \quad (14)$$

(note that $\chi_0^* = 1$ because $\chi_0 = 1$) and *nonuniquely* for the truncated fraction f:

$$f = \chi_m, \quad m \geqslant 2M.$$

[2] The use of a Dirac delta-function at the endpoint $\theta = 0$ of the integration interval, rather than in the interior as is more customary, can be made entirely rigorous (Friedman, 1965, p. 154).

The selection of f thus appears somewhat indeterminate. However, for consistency with the delta-Eddington approximation, and because all our work has led us to believe that it is more important to get lower order moments correctly than higher order ones, we fix on

$$f=\chi_{2M}. \tag{15}$$

This indeterminacy of f corresponds in a limited sense to the infinite variety of ways in which one may truncate the phase function "by eye," as people have done in the past. However, the scope of the indeterminacy is dramatically reduced here. Furthermore, the choice in Eq. (15) seems the most natural in many ways, whereas there was no apparent basis for selection among previously proposed truncation procedures.

The delta-function has the following Legendre polynomial expansion (Morse and Feshbach, 1953, p. 729):

$$\delta(x-y)=\sum_{n=0}^{\infty}\tfrac{1}{2}(2n+1)P_n(x)P_n(y). \tag{16}$$

Using this formula in Eq. (13) with $x=1$ and $y=\cos\theta$, together with Eqs. (3), (14) and (15), shows that the approximate phase function and the exact one differ by

$$P(\cos\theta)-P^*(\cos\theta)=\sum_{n=2M+1}^{\infty}(2n+1)(\chi_n-\chi_{2M})\times P_n(\cos\theta). \tag{17}$$

Thus the exact and approximate phase functions agree in the first $2M$ terms of their Legendre polynomial expansions, so that they obviously become equal in the limit $M\to\infty$. Furthermore, the leading coefficients of the difference are considerably reduced by having χ_{2M} subtracted from them, i.e., $|\chi_n-\chi_{2M}|\ll|\chi_n|$ for these coefficients; a price is paid for this in that the higher terms of the difference (where $\chi_n\approx 0$) are inflated, but these terms are relatively of much less importance since they tend to cancel one another out when $\theta\neq 0$ [Eq. (16) with $x\neq y$ illustrates this cancellation process]. Finally, the difference (17) tends not to grow, and in fact usually diminishes, as $P(\cos\theta)$ becomes increasingly asymmetric, because then the χ_n decrease more slowly and more of the leading coefficients are small.

The azimuthal average [Eq. (8a)] of the approximate phase function (13) is

$$\bar{P}^*(\mu,\mu')=2f\delta(\mu-\mu')+(1-f)\mathcal{P}(\mu,\mu'), \tag{18}$$

where

$$\mathcal{P}(\mu,\mu')\equiv\sum_{n=0}^{2M-1}(2n+1)\chi_n^*P_n(\mu)P_n(\mu'). \tag{19}$$

The second term in Eq. (18) follows in the usual way from the spherical harmonic addition theorem. The first term follows from using Eq. (16) and the addition theorem:

$$\begin{aligned}&\int_0^{2\pi}\delta[1-\mu\mu'-(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi]d\phi\\&=\sum_{n=0}^{\infty}\tfrac{1}{2}(2n+1)\int_0^{2\pi}P_n[\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi]d\phi\\&=\sum_{n=0}^{\infty}\tfrac{1}{2}(2n+1)2\pi P_n(\mu)P_n(\mu')\\&=2\pi\delta(\mu-\mu'),\end{aligned}$$

where the final equality follows from putting $x=\mu$, $y=\mu'$ in Eq. (16).

If Eq. (18) is put into the transfer equation (6), a transfer equation of identical form is obtained, but with $\mathcal{P}$ replacing $\bar{P}$, τ' replacing τ, and ω' replacing ω, where

$$d\tau'=(1-\omega f)d\tau, \tag{20a}$$

$$\omega'=(1-f)/(1-\omega f)\omega. \tag{20b}$$

[These are the correct transformations in the general case; only when ω and f are independent of τ does (20a) reduce to the form given in Joseph *et al.* (1976).] For small values of M, $\mathcal{P}$ is much simpler than $\bar{P}$, and thus the transformed equation of transfer is much easier to solve.

There is no need to renormalize the phase function in the δ-M method. To see this, consider first the doubling method, wherein it is advantageous to pick M Gaussian angular quadrature points $0<\mu_1<\cdots<\mu_M<1$ on the interval $[0,1]$, with mirror-symmetric points on $[-1,0]$, for a total of $2M$ "streams." Then, if the Gaussian weights are $C_1,\ldots,C_M$, the flux conservation condition arising from Eq. (10) [which turns out to be simply the quadratured form of Eq. (11)] is

$$\tfrac{1}{2}\sum_{i=1}^{M}C_i[\mathcal{P}(\mu_i,\mu_j)+\mathcal{P}(\mu_i,-\mu_j)]=1. \tag{21}$$

Replacing $\mathcal{P}$ in this condition by its explicit form (19) leads to

$$\begin{aligned}&\tfrac{1}{2}\sum_{i=1}^{M}C_i\sum_{m=0}^{2M-1}(2n+1)\chi_n^*P_n(\mu_i)[P_n(\mu_j)+P_n(-\mu_j)]\\&=\tfrac{1}{2}\sum_{n=0}^{2M-1}(2n+1)\chi_n^*[1+(-1)^n]P_n(\mu_j)\sum_{i=1}^{M}C_iP_n(\mu_i)\\&=\tfrac{1}{2}\sum_{n=0}^{2M-1}(2n+1)\chi_n^*[1+(-1)^n]P_n(\mu_j)\int_0^1P_n(\mu)d\mu\\&=\chi_0^*P_0(\mu_j)=1.\end{aligned}$$

The sum in the second step is equal to the integral in the third step by the definition of Gaussian quadra-

ture. That integral vanishes for all even $n>0$, while the factor $[1+(-1)^n]$ vanishes for all odd n. Thus only the $n=0$ term contributes; and since $\chi_0^*=1$, the final step follows immediately.

A similar result holds for the discrete ordinate method (Liou, 1973), where, for M Gaussian quadrature angles $-1<\bar{\mu}_1<\ldots<\bar{\mu}_M<1$ on the interval $[-1,1]$, with corresponding weights $\bar{C}_1, \ldots, \bar{C}_M$, the flux conservation condition is

$$\tfrac{1}{2}\sum_{i=1}^{M}\bar{C}_i\mathcal{P}(\bar{\mu}_i,\bar{\mu}_j)=1. \tag{22}$$

Replacing $\mathcal{P}$ by its explicit form (19) and noting that

$$\sum_{i=1}^{M}\bar{C}_iP_n(\bar{\mu}_i)=\int_{-1}^{1}P_n(\mu)d\mu, \quad n\leqslant 2M-1$$

(by the definition of Gaussian quadrature) shows that condition (22) is identically satisfied.

The spherical harmonic method, being essentially identical to the discrete ordinate method for *flux* computations (Irvine, 1975), requires no separate discussion. The proofs that conditions (21) and (22) are true break down if non-Gaussian quadrature angles are employed. This argues strongly for choosing the Gaussian angles.

When the usual diffuse-direct separation is made in the transfer equation (e.g., Joseph *et al.*, 1976), an extra term, involving $\mathcal{P}(\mu,\mu_0)$, is added to the source. Then when the flux conservation condition arising from Eq. (10) is enforced, in addition to Eq. (21), we find a further necessary condition:

$$\tfrac{1}{2}\sum_{i=1}^{M}C_i[\mathcal{P}(\mu_i,\mu_0)+\mathcal{P}(-\mu_i,\mu_0)]=1. \tag{23}$$

But this is automatically satisfied when (21) is, since μ_j in (21) may be replaced by μ_0 without affecting any of the subsequent proof. [The discrete ordinate method analog to condition (23) is likewise identically satisfied if condition (22) is.] In the old ways of renormalizing, (23) was a separate, independent condition to be satisfied, although this was never explicitly noted.

There is no guarantee that $\mathcal{P}>0$ in the δ-M method. We have (only for $M=1$ or 2) observed negative values, but the quality of the results is not affected since $\delta-M$ deals only with *moments*, and not particular values, of the phase function.

b. *The phase function moments*

The viability of the δ-M method rests, in part, on being able to compute the low-order phase function moments $\{\chi_m|m=0,\ldots,2M\}$ economically. We have found an exceedingly simple yet elegant procedure for doing this, which is based on a Lobatto quadrature of Eq. (4) and which even monitors its own error, based on the difference between the computed value of χ_0 and unity. The details are given in the Appendix.

The importance of an economical Mie computation will be grasped immediately by any readers who have had to do such computations. For those who have not, we might note that, using our own Mie program (which has been continually developed and optimized over a four-year period), 73 min of CDC 7600 time was required to compute 200 moments of the Mie phase function of Fig. 1. This involved computing the phase function at 400 Lobatto angles and integrating the size distribution out to a radius of 40 μm using a Mie size parameter step $\Delta x=0.05$ all the way. While this particular calculation used four or five times as many angles as one would normally choose (cf. Table A2 in the Appendix), and achieved an accuracy several significant digits beyond what one would normally aim for, it serves to illustrate the potential voracity of Mie programs. (The moment calculation just described is used in testing the δ-M method in Section 5b.)

A practical moment computation is normally keyed to the desired accuracy in fluxes, which in turn depends on M. There is little point in computing moments to, say, 5D if one only wants fluxes to 2D. Generally the moments need be accurate to just one more digit than the fluxes are expected to be.

In all the examples we have examined, only rarely have we observed deviations from the rule that the moments χ_n are positive, and decrease monotonically as n increases. When these rules are violated, it is often *prima facie* evidence that the moments in question are inaccurate. This monotonic decrease is particularly evident for the Henyey-Greenstein phase function (12), for which $\chi_n=g^n$. But our survey has admittedly been restricted to fairly structureless phase functions (e.g., Mie phase functions for fairly broad size distributions). If there is more structure, so that for example a pronounced peak in the phase function lines up with a peak in a Legendre polynomial P_n, the corresponding moment χ_n may easily exceed its immediate predecessors.

The moments for typical Mie phase functions of asymmetry factor g fall off much more slowly than g^n. But calculated fluxes change by at most a few percent when such phase functions are replaced by Henyey-Greenstein ones with the same g (Hansen, 1969a). This indicates that higher moments must be much less important for the flux than lower ones, which undoubtedly accounts for the excellent accuracy (Section 5) of the δ-M method after correctly matching only the first few moments.

c. *Advantages*

The advantages of the δ-M method are as follows. First, it is highly accurate for small values of M

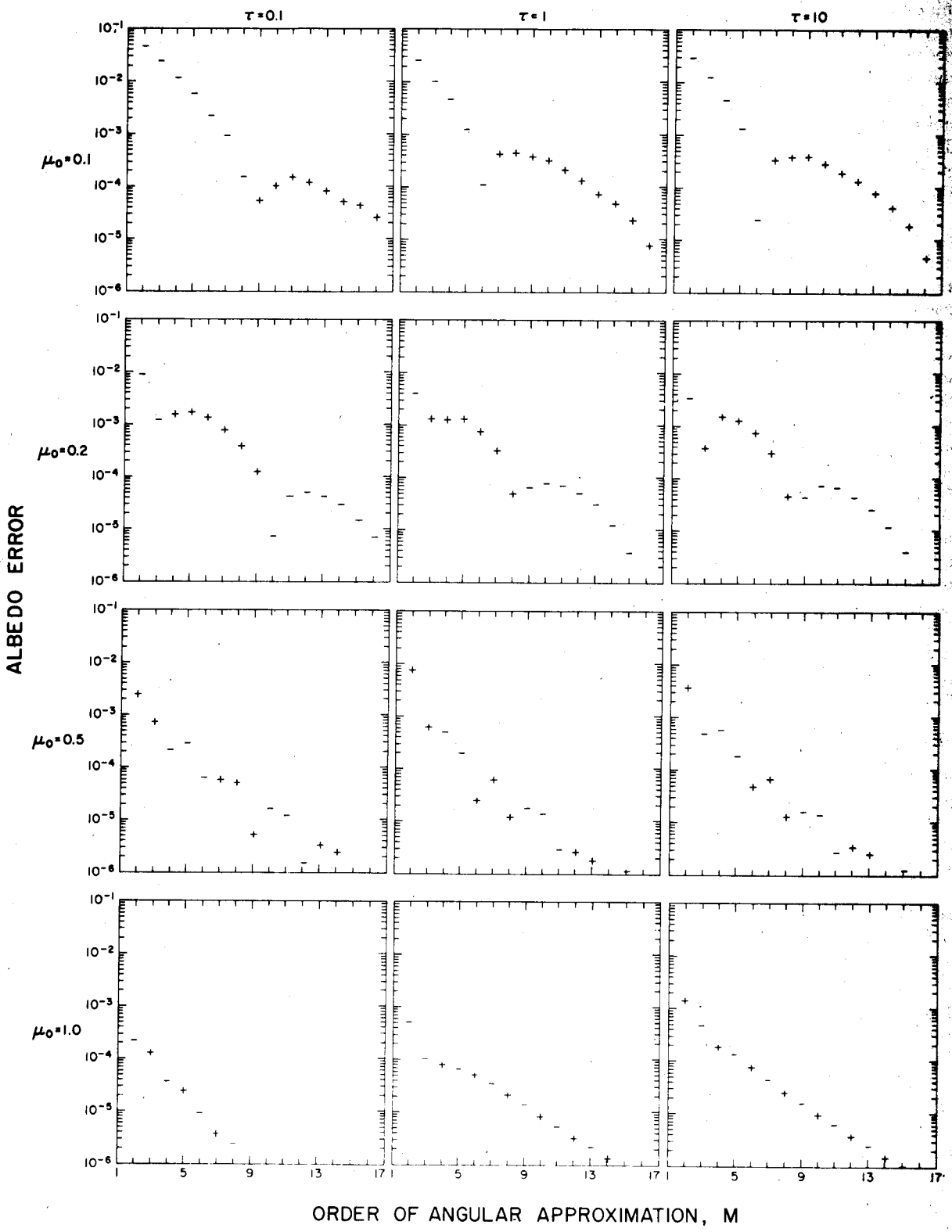

FIG. 3. Absolute albedo error, plotted using plus signs when positive and minus signs when negative, as a function of the order of angular approximation M in the δ-M method. The case considered is a homogeneous layer with $g=0.85$ Henyey-Greenstein phase function, $\omega=0.9$, and zero bottom-boundary albedo.

(e.g., for $M=8$ it can guarantee anywhere from 4–6 significant digits in fluxes). Examples in Section 5 illustrate this point.

Second, it converges rapidly, unlike previous methods reviewed in the Introduction, which will not converge past 2–3 significant digits no matter how far you increase their equivalent of M.

Third, flux conservation is automatic; no *ad hoc* "renormalization" procedures are needed to force it.

Fourth, truncation tends to zero ($f\to 0$) as $M\to\infty$; all previous methods have used fixed truncation, or none at all. Also, the arbitrariness in truncation is to a large extent removed.

Fifth, Mie computations are shortened, since the phase function will generally not need to be calculated at so many angles as previously (see the Appendix). Also, *long* tables of phase function *values* can be replaced by *short* tables of phase function *moments*.

Sixth, the computation of the azimuthally averaged phase function is dramatically simplified—it follows from just a short series [Eq. (19)]. Previously, because getting *all* the moments χ_n required too much computation, the author found the integral form [Eq. (8a)] preferable to the long series [Eq. (8b)]. But numerical quadrature for that integral proved inaccurate and lengthy when the phase function was in tabular form. Interpolation was tricky and could cause great loss of accuracy, as could a too simple quadrature scheme. In all, the integral (8a) was difficult; the code to calculate it was much more complex than that for Eq. (19), and was an order-of-magnitude more time-consuming.

5. Examples and discussion of errors

All the following examples pertain to a single homogeneous layer with zero upwelling radiation at the bottom boundary. The albedo and absorptivity of this layer are computed from the diamond-initialized doubling method (Wiscombe, 1976) using an initial layer of optical depth $\Delta\tau=\mu_1$, where $\mu_1>0$ is the smallest angular quadrature point. [Wiscombe (1977) examines the effect of varying $\Delta\tau$.]

a. Henyey-Greenstein case

Let the layer have a Henyey-Greenstein phase function (12) with $g=0.85$ (as shown in Fig. 1). Let its single-scattering albedo be $\omega=0.9$, so that it is rather highly absorbing. The exact answers are calculated using the $\delta-M$ method with $M=50$; since $f=0.85^{100}=9\times10^{-8}$ for $M=50$, the approximate phase function (13) is essentially exact. Six-figure accuracy or better is attained, as verified by going up to $M=65$ and observing no changes in these figures.

Fig. 3 shows the absolute value of the error in albedo as a function of M, for $2\leqslant M\leqslant 16$. Layer optical depths $\tau=0.1$, 1 and 10 are shown in the columns of Fig. 3, and solar zenith angle cosines $\mu_0=0.1$, 0.2, 0.5 and 1.0 in the rows. Pluses indicate the error is positive, minuses, negative; although it is somewhat unconventional, this representation was deemed best for showing both the *absolute magnitude* of the error (which is of more interest than its signed magnitude) and its oscillatory behavior.

The oscillatory behavior of the error, with an amplitude which declines almost exponentially, is the most striking feature of Fig. 3. The pattern shown is universal; it exhibits only relatively minor changes no matter what phase function is used and no matter what the values of optical depth, single-scattering albedo and surface albedo. [One cannot help but believe that an analytic form for the error—something like $\exp(-\alpha M)\sin\beta M$—must exist; we leave this as an open challenge.] The period of the oscillation is a strong function of μ_0 and is practically independent of everything else. For $\mu_0=0.1$, we see somewhat less than one complete cycle, while there are almost three complete cycles for $\mu_0=0.5$ and seven for $\mu_0=1$. Indeed, for $\mu_0=1$ the error oscillates with maximum rapidity, since it changes from plus to minus each time M increases by 1.

An obvious consequence of the oscillation is that *increasing M may increase the error*. This point is worth emphasizing, for it is a common misperception that increasing the order of an approximation necessarily reduces the error. This is true only in the large—if M is increased *far enough*, the error is reduced. But increasing M by only 2 or 3 or 4 may actually have a detrimental effect on accuracy.

If one actually had an analytical form for the error, which is suggested so strongly by the regularities in Fig. 3, one could use an extrapolation-to-the-limit technique, starting from a few results for small values of M. Note, however, that the pattern may not be firmly established until $M\geqslant 4$.

Rapid convergence is the other striking feature of Fig. 3. By $M=8$ one is already assured of 3–6 digits of accuracy in albedo (and absorptivity and transmissivity as well). This is particularly impressive in view of the fact that $M=8\ll N\approx 86$ (the number of terms in the phase function Legendre polynomial expansion).

The error falls sharply (1–2 orders of magnitude or more) as μ_0 increases; this is typical of all truncation-type methods (cf. Joseph *et al.*, 1976). The smallest value of μ_0 to be considered always determines the accuracy one may expect in flux ratios, like albedo. For fluxes themselves, on the other hand, the errors shown in Fig. 3 are multiplied by μ_0; the flux errors for $\mu_0=0.1$, for example, are reduced by an order of magnitude. The error is virtually independent of τ; it may increase somewhat with τ, though never by as much as an order of magnitude.

Fig. 4 shows the absolute value of the *absorptivity* error as a function of M; in all other respects it is

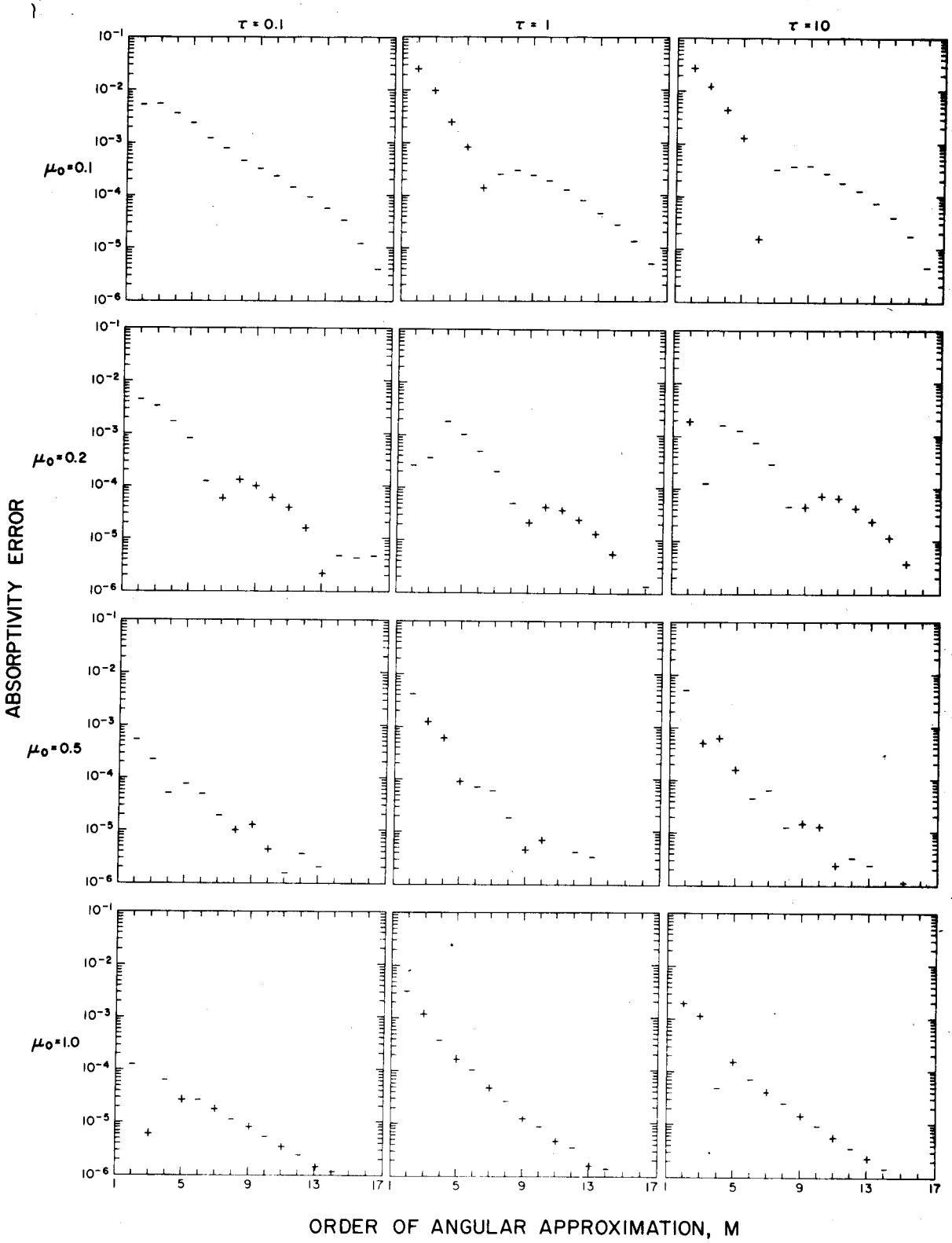

FIG. 4. Absolute absorptivity error vs M, for the same situation as in Fig. 3.

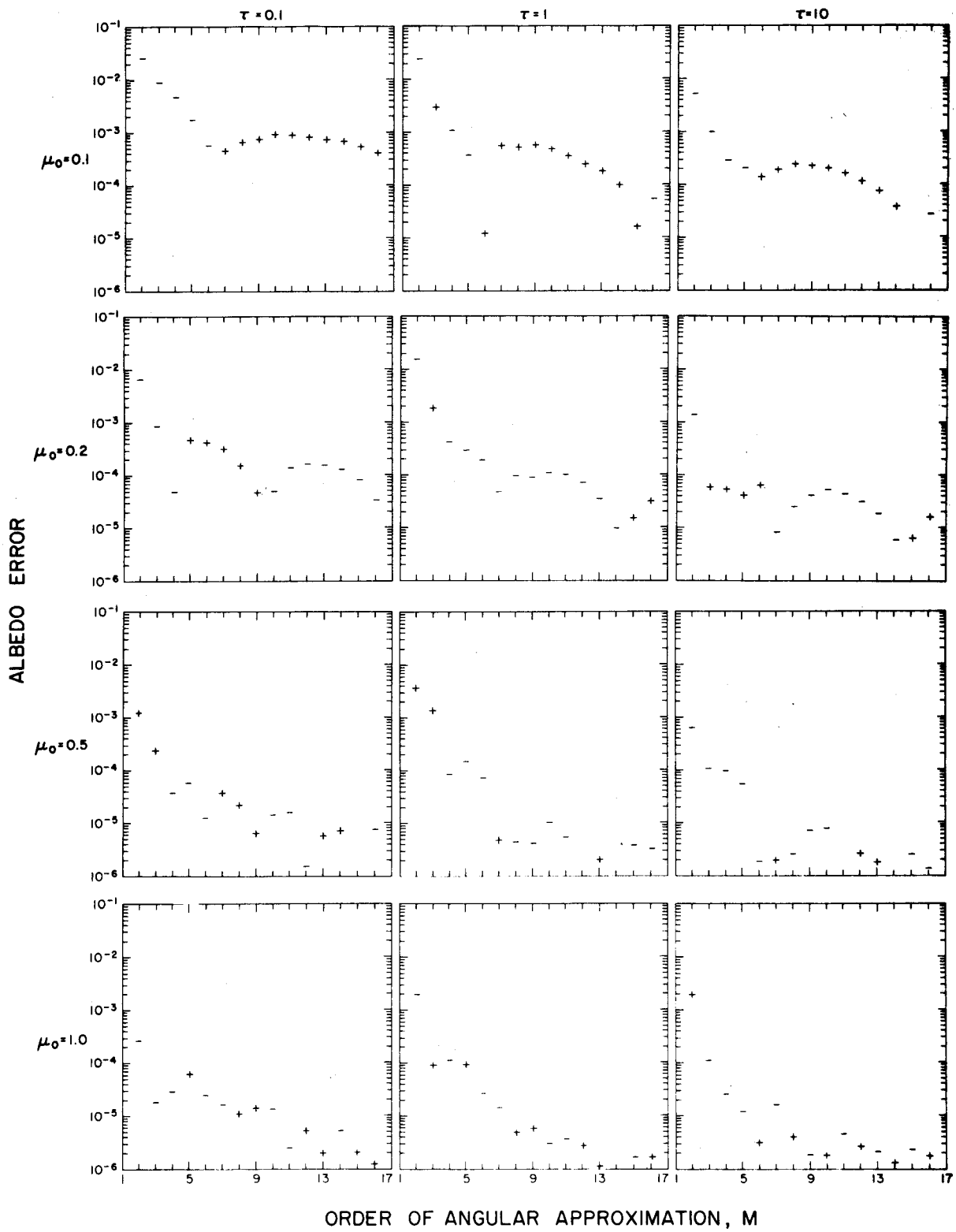

FIG. 5. Absolute albedo error vs M, as in Fig. 3 except that $\omega = 1$ and the layer has the Mie phase function of Fig. 1.

identical to Fig. 3. In general, the absorptivity error is of opposite sign to the albedo error and no more than an order of magnitude different from it. There does not seem to be a uniform tendency for absorptivity error to be smaller than albedo error, as there is in the delta-Eddington approximation.

b. Mie case

Let the layer in question have the Mie phase function shown in Fig. 1 and a single-scattering albedo $\omega=1$. The "exact" answers are provided by the δ-M method with $M=60$, for which the truncated fraction $f=0.18$. Four-figure accuracy is attained for $\mu_0=0.1$, five-figure for $\mu_0=0.2$ and six-figure or better for $\mu_0\geqslant 0.5$, as verified by increasing M in steps all the way up to $M=80$ (where $f=0.11$) and observing convergence to these numbers of figures. We found the $M=60$ solution preferable (to solutions with larger M) because, at large optical depths, it exhibited less spurious absorption due to round-off error accumulation. In spite of the variable accuracy, errors are all plotted on the same scale (10^{-1}–10^{-6}) for ease of intercomparison.

Fig. 5 shows absolute values of albedo errors as a function of M for the same set of τ and μ_0 as in Figs. 3 and 4. Most of the comments made about the Henyey-Greenstein example apply here without change, especially as regards oscillation, rapid convergence and behavior with μ_0. The only modification is that errors *do* tend to fall as τ increases here, though generally by somewhat less than an order of magnitude, and erratically for $M\leqslant 6$.

The primary difference from the Henyey-Greenstein case is that the error here behaves less smoothly as a function of M, undoubtedly reflecting the considerably greater structure in the Mie phase function. For example, for $\mu_0=1$, the monotonic error decrease is absent here, as is the uniform plus-minus error pattern (except when $\tau=10$).

Most surprising is that the errors are not notably larger here compared to the Henyey-Greenstein case, even though the Henyey-Greenstein Legendre polynomial expansion is much shorter and is therefore, one would presume, better approximated by small values of M. The compensating factor in favor of the Mie phase function is, we believe, that its forward peak *looks* much more like a delta-function (see Fig. 1b). The Mie errors do seem to decay more slowly than the Henyey-Greenstein ones as $M\rightarrow\infty$, but for the working range of M in Figs. 3 and 5, this asymptotic separation is not dramatically apparent.

c. Other cases

We have in fact examined many hundreds of plots such as are seen in Figs. 3–5, for several other phase functions and many parameter values. We shall briefly summarize the conclusions from this survey.

The error tends to maximize around a single-scattering albedo $\omega=0.5$ and to fall off a factor of 2–3 in going from there to either $\omega=0.2$ or $\omega=1$. In the range $0.9\leqslant\omega\leqslant 1$, relevant to water clouds in the solar spectrum, the error is almost invariant with ω.

Errors for $\tau=0.01$ tend to be decidedly smaller than for $\tau=0.1$. Albedo error still oscillates at $\tau=0.01$, while absorptivity error does not, becoming instead purely negative and generally monotone decreasing.

Errors at $\tau=100$ are nearly the same as at $\tau=10$ when $\omega\leqslant 0.9$. For ω near unity, errors tend to decrease, though by less than an order of magnitude, in going from $\tau=10$ to $\tau=100$.

As surface albedo increases from 0 to 0.8 the error 1) falls for $\omega=1$, 2) changes little for $\omega=0.5$ and 3) rises for $\omega=0.2$. The rises or falls are smaller, the larger the optical depth, and never exceed an order of magnitude; the error curve tends to shift up or down as a whole in this process.

We conclude from this survey that the error curves in Figs. 3–5 are representative, to within an order of magnitude, of errors for most other highly asymmetric phase functions and all other values of ω, τ and surface albedo.

6. Comparison with discrete ordinate method

Liou (1973) presents five-figure tables of discrete ordinate method (DOM) results for a $g=0.75$ Henyey-Greenstein case and for 2, 4, 8 and 16 streams. This sort of tabulation is extremely rare in the literature, and while the chosen phase function is *not* very asymmetric [$P(\theta=0)=28$; see Fig. 1], we have undertaken to compare δ-M errors with DOM errors for this case.

Table 1 gives ratios of DOM to δ-M albedo errors (the DOM 2-stream case is omitted since it is everywhere dramatically inferior to the delta-Eddington approximation). Entries less than 1 are italicized; in these cases (one-eighth of the total) DOM error is smaller, although typically by no more than a factor of 2. In the great majority of cases δ-M is more accurate, increasingly so as the number of streams increases. Indeed, by 16 streams ($M=8$), the δ-M error is 2–3 orders of magnitude smaller. Its rapid convergence compared to DOM is dramatically highlighted here.

The DOM must introduce further approximations to get upward and downward fluxes, since its quadrature rule is selected to integrate only the *complete* angular range, $\mu\in[-1,1]$. Perhaps this explains the odd behavior of its albedo error, which is smallest for *grazing* ($\mu_0=0.1$) rather than near-normal ($\mu_0=0.9$) incidence. The δ-M method behaves exactly oppositely, which is why the ratios in Table 1 tend to be largest for $\mu_0=0.9$ and smallest for $\mu_0=0.1$. The exceptions to this tendency, and in general the sort of erratic behavior one sees in Table 1, are a result of

TABLE 1. Ratio of DOM (Liou, 1973) to δ-M albedo errors for 4, 8 and 16 streams, for a homogeneous layer having a $g=0.75$ Henyey-Greenstein phase function. Cases in which DOM is more accurate are italicized.

			Number of streams		
ω	τ	μ_0	4	8	16
1.0	0.25	0.1	*0.3*	2	6
		0.5	5	120	110
		0.9	10	20	20
	1	0.1	2	3	100
		0.5	3	*0.4*	30
		0.9	140	90	110
	4	0.1	*0.4*	5	150
		0.5	2	5	140
		0.9	5	10	280
	16	0.1	*0.6*	5	160
		0.5	3	*0.1*	290
		0.9	10	10	1040
0.8	0.25	0.1	*0.4*	2	40
		0.5	2	5	70
		0.9	30	90	280
	1	0.1	1	6	780
		0.5	*0.7*	4	410
		0.9	20	4	180
	4	0.1	1	6	1800
		0.5	*0.7*	4	1030
		0.9	20	2	240
	16	0.1	1	6	1800
		0.5	*0.7*	4	1030
		0.9	20	2	250

the error oscillations of δ-M and DOM being out of phase.

An analog to Table 1 for absorptivity error revealed nothing fundamentally different from what we have already stated.

7. Summary and conclusions

By extending the idea behind the delta-Eddington approximation (Joseph *et al.*, 1976) in a natural way, a technique of great power, which we have dubbed the δ-M method, is created. It is designed to treat highly asymmetric phase functions, and it renders remarkably accurate computed fluxes at very low orders of angular approximation M, no matter how great the asymmetry. $M=8$, or a total of 16 streams, is sufficient for 3–6 digit accuracy in flux ratios (albedo, etc.) and 4–6 digit accuracy in fluxes themselves.

Advantages of the δ-M approximation, besides its high accuracy, are as follows:

1) Flux conservation is automatic—no "renormalization" of the phase function is required.

2) Phase function truncation is put on a much sounder and less *ad hoc* footing, and it goes to zero as $M\rightarrow\infty$.

3) Errors behave in a regular and simply understood fashion.

4) Convergence is rapid as $M\rightarrow\infty$, although, because of its oscillatory behavior, error may *increase* when M increases by small amounts.

5) Necessary Mie computations and associated tables are considerably reduced in size.

More subjectively, the δ-M approximation has a certain elegance which the author found lacking in previous methods for dealing with asymmetry. Also, it renders the passage to the limit $M\rightarrow\infty$ much more transparent than it has been in the past; indeed, as reviewed in the Introduction, previous techniques have had great difficulty converging at all as their equivalent of M tends to infinity.

Comparison of Liou's (1973) discrete ordinate method with δ-M for a phase function of asymmetry factor 0.75 reveals δ-M to be generally equivalent or superior for $M=2$ and 4, and superior by two orders of magnitude for $M=8$. This superiority can only be accentuated for more asymmetric phase functions.

This work has left untouched the problem of the angular dependence of radiative intensities. Clearly, δ-M may be inadequate to this problem for M small, if for no other reason than that a very small number of streams simply cannot resolve the features in something like the bidirectional reflectivity. Nevertheless, barring phenomena such as scattering into an aureole or scattering with near-grazing incidence, where the shape of the forward peak is crucial, we are optimistic that reasonably good angular information can be generated by the δ-M method without going to large values of M. However, we must leave this matter to other investigators.

Acknowledgment. The author would like to thank Dr. Robert Dickinson for allowing him the freedom to pursue this work in the face of pressures to become more "relevant."

APPENDIX

Economical Computation of Phase Function Moments

Computation of integrals is normally fairly trivial, but not when they involve a Mie phase function, as the moment integrals (4) may. To compute a *single* value of the Mie phase function in Fig. 1 takes 10 sec on a CDC 7600 computer, and 100 or more such values may be needed in doing the moment integrals. Furthermore, one set of phase function values must suffice for doing a whole *set* of moment integrals. Any quadrature scheme must be carefully designed

TABLE A1. Accuracy of first 20 moments of Henyey-Greenstein phase functions with $g=0.85$, 0.95, as computed by Lobatto quadrature rules (A1) and (A2). nD means the *least* accurate moment is correctly computed to n decimal places.

	$g=0.85$		$g=0.95$	
L	Eq. (A1)	Eq. (A2)	Eq. (A1)	Eq. (A2)
20	0D	3D	0D	1D
40	3D	8D	0D	5D
60	6D	12D	1D	7D
80	8D	12D	2D	10D

if it is to give reasonably accurate moments without wasting vast amounts of computer time.

The problem is that the integrand in Eq. (4) has both a large forward peak and numerous oscillations. [$P_m(\cos\theta)$ has m zeroes in $\theta\in[0,\pi]$ which are about equally spaced.[3]] A quadrature rule must resolve the

$$\frac{2n-1}{2m+1}\pi\leqslant\theta_n^{(m)}\leqslant\frac{2n}{2m+1}\pi.$$

peak *and* the oscillations, at the same time using a minimal number of quadrature angles.

Hunt (1970) reviews the scanty literature on this subject and proposes

$$\chi_m\approx\tfrac{1}{2}\sum_{l=1}^{L}W_lP(\mu_l)P_m(\mu_l), \qquad \text{(A1)}$$

where μ_l are the abscissas, and W_l the weights for Lobatto quadrature on $[-1,1]$. This method is deficient, however, in that the quadrature angles $\cos^{-1}\mu_l$ do not cluster in the forward peak; they are instead almost equally spaced, e.g., for $L=100$, the first few are 0°, 2.2°, 4.0°, 5.9° and 7.7°.

Consider instead the quadrature formula

$$\chi_m\approx\tfrac{1}{2}\sum_{l=1}^{L}C_lP(\cos\theta_l)P_m(\cos\theta_l)\sin\theta_l, \qquad \text{(A2)}$$

where θ_l are the abscissas and C_l the weights, for Lobatto quadrature on $[0,\pi]$. Here quadrature angles *are* clustered in the forward peak; for $L=100$, the first few θ_l are 0°, 0.07°, 0.22°, 0.47° and 0.81°. Table A1 shows the dramatic superiority of (A2) to (A1) for computing the first 20 moments $\chi_m=g^m$ of Henyey-Greenstein phase functions with $g=0.85$ and $g=0.95$. [Lobatto abscissas and weights were computed from the algorithm of Michels (1963).]

In general, (A2) gives the first L moments, all to about the same accuracy, provided L exceeds some lower bound L_0 which depends on the particular phase function. For Henyey-Greenstein phase functions, with $g\leqslant0.95$, $L_0=30$ gives at least 3D. For cloud phase functions with drop distribution as described in Section 2, but at several wavelengths, L_0 can be deduced from Table A2. Clearly $L_0=61$ gives at least 3D for all the wavelengths. This is surprising on two counts. First, the forward peak resolution is poor at $L=61$; for $\lambda=0.5$ μm, the first few $L=61$ phase function values are 9724, 8908, 4123, 741, 136 and 41. Second, for higher moments like χ_{60}, the 61 quadrature angles barely resolve the oscillations [of $P_{60}(\cos\theta)$] at all. Apparently $\sin\theta_l$ acts as an important mitigating factor in (A2), as far as the forward peak is concerned; and the oscillations do not need to be resolved any better by virtue of using a Gaussian-type quadrature.

The right-hand side of Eq. (A2) for $m=0$ is an excellent error monitor. If it equals unity to k decimal places (remember $\chi_0\equiv1$), then $\{\chi_m|m=1,\ldots,L\}$ will be accurate to between $k-1$ and $k+1$ decimal places.

The success of Eq. (A2) for $L\ll N$ indicates that $P(\cos\theta)\sin\theta$ is a far lower degree polynomial in θ, than $P(\cos\theta)$ is in $\cos\theta$. Why this should be so, and what practical utility it might have, is not obvious, since the natural formulation of radiative transfer is in terms of $\cos\theta$.

Changes in the wavelength in the fourth decimal place can alter computed moments in the same place. Changing the Mie size parameter integration increment

[3] These zeroes $\{\theta_n^{(m)}|n=1,\ldots,m\}$ satisfy (Abramowitz and Stegun, 1965, Eq. 22.16.6)

TABLE A2. Accuracy of first $(L-1)$ moments of Mie phase functions at various wavelengths λ, as computed by Eq. (A2). Drop size distribution is described in Section 2. Liquid water indices of refraction are from Hale and Querry (1973), Palmer and Williams (1974) and Downing and Williams (1975).

λ	0.5 μm	1 μm	2 μm	10 μm
Index of refraction	1.335	$1.328-3.35\times10^{-6}i$	$1.306-1.16\times10^{-3}i$	$1.220-0.0515i$
g	0.86	0.85	0.84	0.91
$P(0°)$	9724	2475	678	60
ω	1.0	0.9996	0.94	0.68
N	902	453	230	50
L				
21	0D	1D	1–2D	4D–6D
41	1–2D	3–4D	4–5D	⩾6D
61	3D	3–5D	5–6D	⩾6D
81	3–5D	4–6D	5–7D	⩾6D
101	4–6D	4–6D	⩾6D	⩾6D
201	5–6D	5–7D	⩾6D	⩾6D

from $\Delta x=0.05$ to $\Delta x=0.1$ changed computed moments for $\lambda=0.5$ μm in the fourth place. Hence there seems little point in computing moments to much better than 3D.

For polydispersions where the Mie size parameter never exceeds about 10, the Hunt rule (A1) is actually preferable to (A2), in that it can give better accuracy with smaller L. This is particularly apparent near the Rayleigh limit, where the phase function is nearly quadratic in $\cos\theta$ but is a much higher degree polynomial in θ. Such cases have not, however, ever involved significant amounts of computation.

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