Appendix F

Elementary Concepts

AF.1 Coordinate Systems

We are interested in specifying the coordinates of the unit propagation vector $\hat{\Omega}$ in both the *Cartesian coordinate system* and the *spherical-polar coordinate system* (see Fig. 3.3). The spherical-polar system defines $\hat{\Omega}$ in terms of the two angles, θ and ϕ. The rectangular system defines $\hat{\Omega}$ in terms of its three projections in the (x, y, z) directions, Ω_x, Ω_y, and Ω_z. The relationships between these two sets of coordinates are

$$\Omega_x = \sin\theta\cos\phi; \qquad \Omega_y = \sin\theta\sin\phi; \qquad \Omega_z = \cos\theta \tag{F.1}$$

where $0 \leq \phi \leq 2\pi$, and $0 \leq \theta \leq \pi$.

AF.2 The Dirac Delta-function

A concept which is useful in the mathematical representation of unidirectional or *collimated light* is the *Dirac δ-function.* This 'function' has the peculiar property that it is zero for finite values of its argument, and unbounded (infinite) when the argument of the δ-function is zero, that is

$$\delta(x) = 0 \qquad (x \neq 0) \qquad \text{and} \qquad \delta(x) \to \infty \qquad (x \to 0). \tag{F.2}$$

Furthermore, the 'area' under the function is unity, that is, it is *normalized*

$$\begin{aligned} \int_a^b dx\delta(x) &= 1 \qquad \text{if a and b are of different sign.} \\ &= 0 \qquad \text{if a and b are of the same sign.} \end{aligned} \tag{F.3}$$

It is possible to define the δ-function for a *vector* argument. If we want to represent the electric field from a concentrated 'source' of unit strength (for example, an electron) at the point $\vec{r} = \vec{r}_0$, we write $\delta(\vec{r}-\vec{r}_0)$. In rectangular coordinates $\delta(\vec{r}-\vec{r}_0)$ can be defined as a product of one-dimensional δ-functions, that is

$$\delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \tag{F.4}$$

The integral properties analogous to those in eqn. (F.3) are

$$\int\int\int d^3\vec{r}\,\delta(\vec{r}-\vec{r}_0) = \int dx \int dy \int dz\;\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = 1 \tag{F.5}$$

when the integration domain includes $\vec{r}_0$. The integral in eqn. (F.5) is zero if the integration domain does not include $\vec{r}_0$.

In spherical polar coordinates we represent

$$\delta(\vec{r}-\vec{r}_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi-\phi_0)\delta(r-r_0). \tag{F.6}$$

The volume element in spherical coordinates is $dV = dAdr = r^2 dr \sin\theta d\theta d\phi = -r^2 dr d(\cos\theta)d\phi$. dA is the element of area normal to $\vec{r}$. The normalization property is

$$\begin{aligned}\int dV\delta(\vec{r}-\vec{r}_0) &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\;\sin\theta \int_0^{r_m} r^2\delta(\vec{r}-\vec{r}_0)dr = 1 \quad (r_m > r_0) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{F.7}) \\ &= 0 \quad (r_m < r_0).\end{aligned}$$

r_m is the (arbitrary) radius of a spherical volume centered at the origin.

A very important property applies to the integral of the product of the δ-function with an arbitrary function, say f. For example, if $f = f(x,y)$, then

$$\int dx \int dy f(x,y)\;\delta(x-x_0)\delta(y-y_0) = f(x_0,y_0). \tag{F.8}$$

It must be kept in mind that the volume of integration must include the 'source point' (x_0, y_o) of the δ-function for eqn. F.8 to apply(otherwise the result is zero).

The one-dimensional δ-function has the units of $(\text{length})^{-1}$, while $\delta(\vec{r}-\vec{r}_0)$ has the units of $(\text{length})^{-3}$. Other mathematical forms of the δ-function in terms of the solid angle are given in the next section.

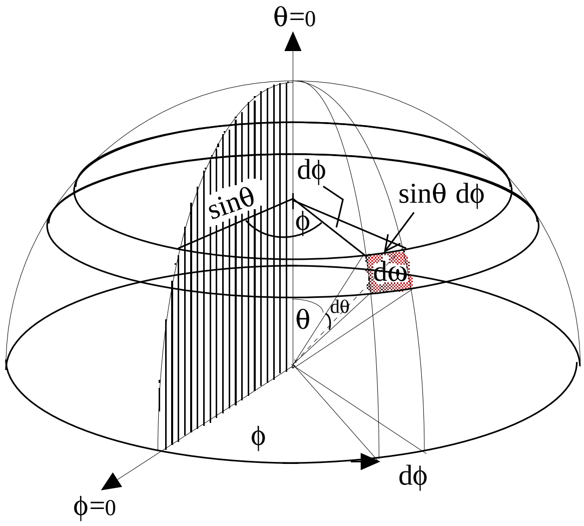

Fig. AF.1. Definition of the solid angle element $d\omega = \sin\theta d\theta d\phi$.

AF.3 The Solid Angle

The *solid angle* ω is defined as the ratio of the area A cut out of a spherical surface (see Fig. AF.1) to the square of the radius of the sphere, i. e. $\omega = A/r^2$. The units of ω are *steradians* $[sr]$. There are 2π sr in a hemisphere, and 4π sr in a full sphere. We are usually interested in a small (differential) element of solid angle, $d\omega$. As shown in Fig. AF.1, $d\omega$ is expressed in spherical-polar coordinates as $d\omega = dA/r^2$. Since $dA = r^2 \sin\theta d\theta d\phi$

$$d\omega = \sin\theta \; d\theta \; d\phi. \tag{F.9}$$

The integral of eqn. F.9 over the sphere, that is over 4π steradians, is

$$\int_{4\pi} d\omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi. \tag{F.10}$$

Often we consider a solar *beam* (§2.2) of light travelling in a particular direction. This direction is called the *propagation direction* and is specified by a unit vector $\hat{\Omega}_0$, which points in the direction (θ_0, ϕ_0). If we consider a general direction, described by the unit vector $\hat{\Omega}(\theta, \phi)$, a beam is a radiative energy flow which is zero for all directions except $\hat{\Omega}_0$. Thus, we can use a *two-dimensional* δ-function $\delta(\hat{\Omega} - \hat{\Omega}_0)$ to specify this energy flow. In spherical-polar coordinates we have

$$\delta(\hat{\Omega} - \hat{\Omega}_0) = \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0) \tag{F.11}$$

where $\hat{\Omega}_0$ is specified by the angles (θ_0, ϕ_0). The normalization property of the δ-function in eqn. F.11 is

$$\int_{4\pi} d\omega \delta(\hat{\Omega} - \hat{\Omega}_0) = 1. \tag{F.12}$$

While $\delta(\hat{\Omega} - \hat{\Omega}_0)$ is non-dimensional, it will be convenient to think of it as having the 'unit' of inverse steradians $[sr^{-1}]$.

Appendix G

A Primer on Absorption and Scattering Opacity

One of the two fundamental properties of light-matter interaction is *absorption*, wherein light energy disappears, and a like amount of energy is converted to heat. The other property is *scattering*, in which the path of the light ray is merely deflected by the matter. We might think that specular reflection from a polished surface is a third type, but this phenomenon can be shown to be a consequence of scattering. Thus two (and only two fates) await a photon when it suffers an encounter with matter. This is true regardless of the form the matter takes: whether in a solid (land surfaces), in condensed form (the ocean) or whether it is composed of gaseous molecules or suspended particles (atmospheres). This book concerns itself with the dual influences of absorption and scattering on radiation fields in planetary media.

Consider first the property of absorption, and imagine a medium in which only absorption is important for the light field. Although it is inherently easier to understand than scattering, it is difficult to find many commonplace examples in which *only* absorption is present. Carbon soot is perhaps the best example. An object covered with soot approaches the ideal *blackbody* behavior, described in elementary thermodynamic textbooks. However, since we are interested in atmospheres and oceans in this book, let us first consider a medium consisting of finely-dispersed soot particles.

Imagine sunlight to fall on such a medium, and consider the attenuation of the light as it passes through this soot cloud. The ability of the medium to attenuate the light will depend upon three quantities: (1) the number per unit volume n of the soot particles; (2) the particle sizes, r; and (3) the distance along the light ray, s. For simplicity we assume the particles are all the same size, and the cloud has uniform spatial density. The relevant attenuation quantity depends upon the *projected cross-sectional area* of the soot cloud in the direction of the light ray, $n\pi r^2 s$. This quantity is a pure number, and is the absorption *opacity*, or *optical depth*, τ_a. (Here we have assumed that the particles act as simple geometric light obstacles, which applies for sizes much larger than the sensing wavelength.) Another way to think of τ_a is the projected shadow area, per unit area, of all the particles along a ray path. If we ignore mutual shadowing effects (and this is usually permissible) a moments thought reveals that the actual distribution of particles along the light ray is unimportant, only the product ns. Thus the relevant quantity is the total *column number* per unit area $\mathcal{N}$, and $\tau = \pi r^2 \mathcal{N}$. A high opacity at a particular visible-light frequency ν means that sunlight will be absorbed high up in the atmosphere, and a small opacity means that it will penetrate deeply. If $\tau(\nu) << 1$, the atmosphere is said to be *optically-thin*, or *transparent* at that frequency, and if $\tau(\nu) >> 1$, it is said to be opaque.

It remains to determine the degree to which the light is transmitted, and this involves a function of τ. It is shown in Chapter 2 that for sufficiently small frequency intervals $\Delta\nu$, this function is the exponential function $\exp[-\tau(\nu)]$. This familiar relationship is known popularly as *Beer's Law*, but for our own reasons, we call it the *Extinction Law.* Since absorption and transmission are opposite sides of the coin, the absorption varies as $1 - \exp[-\tau(\nu)]$. The absorption process leads to a heating of the particles, in contrast to the scattering process.

Atmospheres also emit their own radiation, as do all bodies whose temperatures are above absolute zero. The solar atmosphere, due to its high temperature, emits copiously in the visible spectrum, whereas the cooler atmospheres of the earth and planets emit most of their energy in the thermal infrared. The opacity also plays a key role in the ability of media to emit radiation. This is one of many examples of the principle of detailed balance which are considered in this book, and is more familiar as *Kirchoff's Law*, which says in brief, that *an efficient absorber is an efficient emitter.* To be more precise, the ability of an atmosphere to emit depends upon its opacity per unit length, or per unit volume, and depends upon the local absorptive properties of the medium.

Scattering processes add complexity to the above situation, in redirecting and modifying the radiation field without destroying it. Even soot particles are not "mini-black holes", but scatter a small amount of light. Otherwise we would not be able to distinguish soot particle texture or color. If the particles were non-scattering, the soot cloud would be invisible, except when viewing the light beam directly — it would behave like a neutral density filter which progressively dims the light as we move farther away from the light source.

Now consider the opposite extreme of finely-dispersed water droplets (fog), which are efficient scatterers of visible radiation. "Reflection" from a cloud of these particles causes an incident light beam to be attenuated in a very similar way to the soot cloud, according to the scattering opacity τ_s. However, the light is not destroyed (or at least only a small fraction) but only deflected from its original path. For example around a fog-enshrouded lampost we witness this process as a host of twinkling starlike points of light. In the original direction of the light, the effect is the same as absorption, that is, a dimming of the light in proportion to the number of scattering particles along the path. The opacity is calculated in exactly the same way, except that the physical process is not a heating of the particles, as in absorption†. In fact a measurement of the attenuation with an ideal detector of small acceptance angle in the two cases of an absorbing soot cloud and a totally-scattering water fog would be exactly the same. This assumes that they have the same opacity. Furthermore if we were to measure the radiation in directions away from the light source, the scattering fog would be a source of secondary 'emission'. The same measurement for the soot cloud would register zero radiation. This secondary light source is due to scattering of the light into our line of vision, and is the reason why we can "see" the cloud itself – for that matter, it explains why we are able to view the world around us. A major complexity in a quantitative description of the scattered light is the fact that every particle "sees" not only the original light source, but also the light scattered from its neighbors. This gives rise to higher orders of scattering, referred to as *multiple scattering*, and this "diffusion" of the light tends to produce a more uniform spatial distribution of brightness. Multiple scattering is one of the important subjects of this book.

Consider some implications of the scattering and absorption/emission processes on the earth's atmosphere and ocean. First,

† Actually, the process of scattering *does* alter the velocity of the particles through a momentum exchange with the incident photons, and strictly speaking, this could cause a heating of the gas. However, these radiation pressure effects are negligible for radiation energies of concern in this book.

because of the atmosphere's high transparency in the visible spectrum ($0.4 - 0.7\mu m$), the earth's land and ocean surfaces are subjected to mostly direct solar heating on cloud-free days. On cloudy, overcast days the light field consists of diffuse (multiply-scattered) photons. In general, both effects provide the so-called *short-wave radiative forcing* of the climate system. At the same time, the land and ocean radiate infrared radiation to the atmosphere, and to space (depending upon the infrared opacity as a function of wavelength). This gives rise to radiative cooling, i.e. *long-wave radiative forcing.* The combined radiative effects, when averaged over the diurnal cycle, lead to a *net radiative forcing*, which is variable over the earth's surface. Spatial and temporal variations in this forcing give rise to weather and climate, which themselves alter the radiative forcing, in a highly non-linear interactive system (called feedback). Long-term changes in the long-wave forcing, such as carbon dioxide increases, will alter the atmosphere and ocean in ways which we do not yet fully understand.

In conclusion, absorption and scattering give rise to attenuation according to the same basic formula, $\exp(-\tau)$. If both processes are present, and this is always the case in the real world, the net opacity is found to be the *sum* of the absorption and scattering opacities, $\tau = \tau_a + \tau_s$. Absorption tends to destroy the radiation field, and heat the absorbing particles. Because of their finite temperature, the particles also radiate light into all directions, in proportion to their absorptive properties as a function of frequency. Scattering redirects an original beam of light into generally all 4π sterradians. Multiple scattering causes the radiation field to become more uniform (diffuse). These two processes give rise to short-wave and long-wave radiative forcing of climate, as well as many other atmospheric phenomena. In this book, we will deal with the "up-front" radiative processes, essential to understanding climate and climate change.

Appendix H

Electromagnetic Radiation: The Plane Wave

We review in this appendix some basic aspects of light. We use *light* as a shorthand for *electromagnetic radiation*, and do not mean to imply *visible light*, which occupies only a small portion of the electromagnetic spectrum. Some simple mathematical fundamentals are provided in Appendix F, including a discussion of elementary concepts such as coordinate systems, the Dirac delta-function, and the solid angle. In this section we restrict our attention to a review of the plane wave, and its polarization properties. More advanced topics concerning the Stokes vector representation, partial polarization and the Mueller matrix are described in Appendix I.

AH.1 Plane Electromagnetic Waves

Light is an electromagnetic phenomenon, along with gamma-rays, x-rays, and radio waves. It is described by solutions of the famous set of equations of J. C. Maxwell, formulated in 1865. These equations in differential form and in *mksa* units for an isotropic, homogeneous, source-free medium, are

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \text{ (a)}; \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \text{ (b)};$$

$$\nabla \cdot \vec{H} = 0 \text{ (c)}; \qquad \nabla \cdot \vec{E} = 0 \text{ (d)}. \tag{H.1}$$

$\nabla\times$ and $\nabla\cdot$ denote the curl and divergence operators, respectively.†

† The relationships between the electric and magnetic field quantities, and the medium properties are called the *constitutive relations*, and are included in the equation set, H.1. See Stratton, J.

$\vec{E}$ and $\vec{H}$ are the *electric* and *magnetic fields*, and t is time. ϵ is the *permittivity*, σ is the *conductivity*, and μ is the *magnetic permeability*, all properties of the medium. A net charge of zero throughout the medium is assumed. The basis of these equations and the medium properties are described in various texts.

A solution of these coupled partial differential equations is sought for this source-free case in which both $\vec{E}$ and $\vec{H}$ are functions of a single spatial variable, and time. Let us assume a purely *dielectric medium*, for which the conductivity σ is zero. Taking the curl† of eqns. H.1a and H.1b and using the vector identity $\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$ together with eqns. H.1c and H.1d, we find that both $\vec{E}$ and $\vec{H}$ satisfy the same second-order wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0; \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{H} = 0 \quad \text{where } c = \frac{1}{\sqrt{\mu\epsilon}}. \tag{H.2}$$

c is the speed of propagation in the medium. In a vacuum, the speed of light is $c_o = 1/\sqrt{\mu_o \epsilon_o} = 2.9979 \times 10^8\ [m \cdot s^{-1}]$. The subscript o denotes the vacuum value. It can readily be shown that *plane waves* of the form

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \Re\left\{\vec{E}_0 e^{i(k\hat{\Omega}\cdot\vec{r} - \omega t)}\right\} \\ \vec{H}(\vec{r}, t) &= \Re\left\{\vec{H}_0 e^{i(k\hat{\Omega}\cdot\vec{r} - \omega t)}\right\} \end{aligned} \tag{H.3}$$

are solutions of eqn. H.2. Here $i = \sqrt{-1}$ is the imaginary unit, $\Re$ denotes the real part, and $\vec{E}_0$ and $\vec{H}_0$ are complex constant vectors. The unit vector $\hat{\Omega}$ points in the propagation direction of the plane wave. $k = \omega/c$ is the *wavenumber* [cm^{-1}], and ω is the *angular frequency* [$\text{rad}\cdot\text{s}^{-1}$], related to the ordinary frequency ν, [$\text{cycles}\cdot s^{-1}$] or [$Hz$], by $\omega = 2\pi\nu$. These solutions are called plane waves because at any fixed time t they have the same value at each point in any plane normal to $\hat{\Omega}$, i. e. at any fixed time t, $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ are constant vectors in each plane defined by $\hat{\Omega} \cdot \vec{r} = \text{constant}$.

Note that we have restricted our attention to harmonic plane waves having a sinusoidal variation in time and space. According to eqn. H.3, each Cartesian component of $\vec{E}$ and $\vec{H}$ will be of the general form (with

A., *Electromagnetic Theory*, McGraw-Hill Book CO., New York, 1941. Jackson, J. D. *Classical Electrodynamics*, New York, Wiley, 1975. A good modern text is Griffiths, D. J., *Introduction to Electrodynamics*, Prentice-Hall, 1981.

† For readers unfamiliar with vector analysis, see for example, Edwards, J. and D. E. Penney, Calculus and Analytic Geometry, 3rd ed., Prentice-Hall, Englewood CLiffs, N.J., Chapter 17.

j denoting either x, y, or z)

$$
\begin{aligned}
E_j(\vec{r}, t) &= e_j \cos(k\hat{\Omega} \cdot \vec{r} - \omega t + \delta_j) \\
H_j(\vec{r}, t) &= h_j \cos(k\hat{\Omega} \cdot \vec{r} - \omega t + \phi_j)
\end{aligned} \tag{H.4}
$$

where e_j and h_j are arbitrary real coefficients, and δ_j and ϕ_j are arbitrary phase angles.

The harmonic plane waves in eqns. H.3 are solutions of the wave equation H.2 for arbitrary values of $\vec{E}_0$ and $\vec{H}_0$. But these solutions must also satisfy Maxwell's equations. Substituting eqns. H.3 in eqns. H.1a and H.1b (with $\sigma = 0$), we find that

$$
\sqrt{\mu}\, \hat{\Omega} \times \vec{H}_0 = \sqrt{\epsilon}\, \vec{E}_0; \qquad \sqrt{\epsilon}\, \hat{\Omega} \times \vec{E}_0 = \sqrt{\mu} \vec{H}_0 \tag{H.5}
$$

from which it follows that $\vec{E}_0 \cdot \vec{H}_0 = 0$, and that both $\vec{E}_0$ and $\vec{H}_0$ are orthogonal to the propagation direction $\hat{\Omega}$. In other words, $\vec{E}_0$, $\vec{H}_0$, and $\hat{\Omega}$ form a right-handed triad.

If we now choose the coordinate system such that $\hat{\Omega}$ is along the positive z-axis, we can write

$$
\vec{E} = \vec{E}_{||} + \vec{E}_{\perp}; \qquad \vec{E}_{||} = E_{||}\hat{e}_{||}; \qquad \vec{E}_{\perp} = E_{\perp}\hat{e}_{\perp} \tag{H.6}
$$

$$
\vec{H} = \vec{H}_{||} + \vec{H}_{\perp}; \qquad \vec{H}_{||} = \sqrt{\frac{\epsilon}{\mu}}\hat{e}_z \times \vec{E}_{||}; \qquad \vec{H}_{\perp} = \sqrt{\frac{\epsilon}{\mu}}\hat{e}_z \times \vec{E}_{\perp}. \tag{H.7}
$$

Here each of the components $E_{||}$, $E_{\perp}$, $H_{||}$, and $H_{\perp}$ satisfies the wave equation, and $\hat{e}_{\perp}$, $\hat{e}_{||}$, and $\hat{e}_z$ are unit vectors forming a right-handed triad

$$
\hat{e}_{\perp} \cdot \hat{e}_{||} = \hat{e}_{\perp} \cdot \hat{e}_z = \hat{e}_{||} \cdot \hat{e}_z = 0, \qquad \hat{e}_{\perp} \times \hat{e}_{||} = \hat{e}_z. \tag{H.8}
$$

$E_{||}$ and $E_{\perp}$ are the electric field components parallel and perpendicular to a plane which contains the z-axis, and whose orientation is otherwise arbitrary.†

From eqns. H.3 and H.6–H.7, it follows that

$$
E_{||} = \Re\{\mathcal{E}_{||}\}; \qquad E_{\perp} = \Re\{\mathcal{E}_{\perp}\}. \tag{H.9}
$$

where the complex amplitudes $\mathcal{E}_{||}$ and $\mathcal{E}_{\perp}$ are given by

$$
\mathcal{E}_{||} = a_{||} \exp[i(kz - \omega t + \delta_{||})] \tag{H.10}
$$

$$
\mathcal{E}_{\perp} = a_{\perp} \exp[i(kz - \omega t + \delta_{\perp})]. \tag{H.11}
$$

† This plane will become the plane of incidence when we consider interactions with interfaces, and the scattering plane when we consider interactions with scattering particles.

Here $a_{||}$ and $a_{\perp}$ are the electric field *amplitudes* and $\delta_{||}$ and $\delta_{\perp}$ are the *phase angles*. Similar forms can be derived for the magnetic components.

We define the wave number in a vacuum, $k_o \equiv \omega/c_o \equiv 2\pi/\lambda_o$, where λ_o is the *vacuum wavelength*. Then we can express eqns. H.9–H.11 in a more convenient form

$$E_{||,\perp} = \Re\left\{a_{||,\perp} \exp\left\{i\left[(k_o mz - \omega t) + \delta_{||,\perp}\right]\right\}\right\} \qquad \text{(H.12)}$$

where $m \equiv c_o/c = \lambda_o/\lambda = k/k_o = \sqrt{\epsilon_o\mu_o/\epsilon\mu}$ is the *index of refraction* of the medium, the ratio of the propagation speed *in vacuo* to that in the medium.† These solutions apply to an ideal *harmonic*, *monochromatic* (single frequency) plane wave of infinite spatial extent $(-\infty < x, y, z < +\infty)$ traveling in the positive z-direction. m is often written as a complex quantity, $m \equiv m_r + im_i$. The value of m_r varies slightly with frequency in natural media: in air it is very close to unity – for example, m_r $(\lambda = 1\ \mu m) = 1.0 + 2.892 \times 10^{-4}$. In pure water, m_r $(\lambda = 486\ nm) = 1.3371$.

The solution for a conducting medium $(\sigma \neq 0)$ is worked out in Problem H.1. In this case, the wave is *damped* or *attenuated* along the propagation direction. The solution can be expressed mathematically in the same form as eqns. H.12. In this case the appearance of a 'damping factor' $\exp(-k_o m_i z)$ multiplying eqn. H.12 shows that the presence of a finite conductivity is associated with *absorption* along the wave direction.

AH.2 Energy Transfer

Light waves transmit energy. It is this feature that makes it possible to detect light away from sources, and it explains how the sun warms the earth and ultimately sustains life. The rate at which energy is transported by light is expressed by the *Poynting vector* $\vec{S}$. This quantity is related to the electric and magnetic field vectors, $\vec{E}$ and $\vec{H}$ through $\vec{S} = \vec{E} \times \vec{H}$. This expression gives both the magnitude and direction of instantaneous energy flow. In other words, $\vec{E} \times \vec{H}$ is the *radiative power* per unit area carried along the wave direction.

For time-harmonic plane-wave solutions it follows from eqns. H.6–H.9

† There are actually *two* light speeds to consider: the *phase speed*, $v_p = c = \omega/k$, and the *group speed*, $v_g = \partial\omega/\partial k$. Since $k = n(\omega)\omega/c_0$, and $m(\omega)$ is generally a function of frequency, ω (that is to say, the medium is *dispersive*) then $v_p \neq v_g$. However in a *non-dispersive medium*, $v_p = v_g$.

that

$$\vec{S} = \vec{E} \times \vec{H} = \sqrt{\frac{\epsilon}{\mu}}\,[E_{||}\hat{e}_{||} + E_{\perp}\hat{e}_{\perp}] \times [E_{||}\hat{e}_z \times \hat{e}_{||} + E_{\perp}\hat{e}_z \times \hat{e}_{\perp}]$$

$$= \sqrt{\frac{\epsilon}{\mu}}\,[E_{||}E_{||} + E_{\perp}E_{\perp}]\hat{e}_z = \sqrt{\frac{\epsilon}{\mu}}\,[\Re(\mathcal{E}_{||})\Re(\mathcal{E}_{||}) + \Re(\mathcal{E}_{\perp})\Re(\mathcal{E}_{\perp})]\hat{\Omega}_0. \quad \text{(H.13)}$$

Here $\hat{\Omega}_0$ is the propagation vector of the wave. We are seldom interested in the instantaneous value of $\vec{S}$. Of greater interest is the *time-averaged value*

$$\langle \vec{S} \rangle = \frac{1}{\langle t \rangle} \int_0^{\langle t \rangle} dt \vec{S}(t) \quad \text{(H.14)}$$

where $\langle t \rangle$ is the averaging time. For a periodic function, $\langle t \rangle$ is an integral number of *wave periods*, where one period is $1/\nu$. It is shown in Problem H.2 that the time average of the product of two time-harmonic functions of the same periodicity is

$$\langle \Re\{a(t)\} \cdot \Re\{b(t)\} \rangle = \frac{1}{2}\Re\{ab^*\} = \frac{1}{2}\Re\{a^*b\} \quad \text{(H.15)}$$

where $a(t)$ and $b(t)$ both are of the form in eqns. H.10 and H.11. The asterisk denotes complex conjugation. Using this result in eqn. H.13, we find that the flow in the general direction $\hat{\Omega}$ is

$$\langle \vec{S} \rangle = \frac{m}{2\mu c_o} \left\{ \frac{\epsilon}{2}[\mathcal{E}_{||} \cdot \mathcal{E}_{||}^* + \mathcal{E}_{\perp} \cdot \mathcal{E}_{\perp}^*] \right\} \delta(\hat{\Omega} - \hat{\Omega}_0) \quad \text{(H.16)}$$

where we have used $c = 1/\sqrt{\mu\epsilon}$ and $m = c_o/c = \sqrt{\mu\epsilon/\mu_o\epsilon_o}$. The quantity in the curly brackets is the energy density $\mathcal{U} = \mathcal{U}_e + \mathcal{U}_m$ of the plane electromagnetic wave, consisting of the sum of electric field ($\mathcal{U}_e$) and magnetic field ($\mathcal{U}_m$) energy densities. Eqn. H.16 shows that the energy density of the plane electromagnetic wave propagates with velocity $c = c_o/m$ in the z-direction.

Also, eqn. H.16 shows that a plane electromagnetic wave may be considered to have two components

$$I_{||} = (m/2\mu c_o)|\mathcal{E}_{||}|^2 \quad \text{and} \quad I_{\perp} = (m/2\mu c_o)|\mathcal{E}_{\perp}|^2. \quad \text{(H.17)}$$

$I_{||}$ and $I_{\perp}$ are called the *intensity components*.† Eqn. H.16 tells us that

† Here we are using the physicists definition of intensity. In fact this is closer to our definition of the flux, or irradiance (Chapter 2).

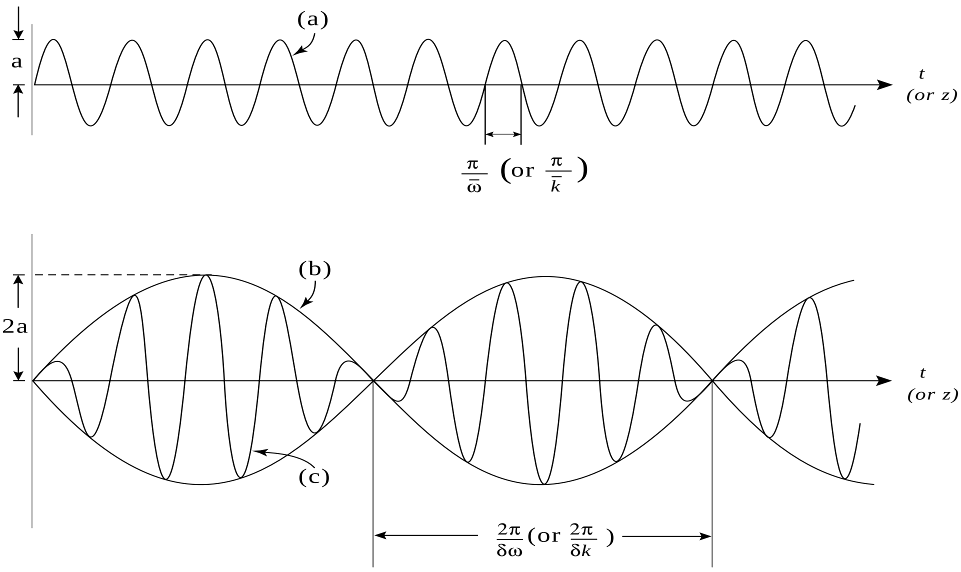

Fig. AH.1. A simple wave packet. (a) The wave $a\cos(\overline{k}z - \overline{\omega}t)$. (b) The wave packet $2a\cos\left[\frac{1}{2}(z\delta k - t\delta\omega)\right]\cos(\overline{k}z - \overline{\omega}t)$. The ordinate represents one of the two independent variables (t or z) while the other is kept constant.

the average radiative power is $(I_{||} + I_{\perp})\ \delta(\hat{\Omega} - \hat{\Omega}_0)$. The fact that light waves have two independent components accounts for the phenomenon of *polarization*. It distinguishes light waves from *scalar waves*, such as sound waves in liquids or gases, which have only a single energy-carrying component.

AH.3 Addition of Plane Waves

The monochromatic plane wave solutions are *elementary* solutions to Maxwell's equations. Clearly, they are idealizations. Any real wave is a linear superposition of monochromatic plane waves of different frequencies, directions, and phases. If all waves in a group have almost the same frequency, we have a *wave packet.* Consider a wave packet consisting of only two waves, both propagating along the z-axis, and having slightly different frequencies and wave numbers. Let the waves have the same amplitude $a_{||}$, and consider only one polarization component, say $E_{||}$. The total electric field is the coherent sum of the individual waves, i. e.

$$E_{||}(z,t) = \Re\left\{a_{||}e^{i(kz-\omega t+\delta_1)} + a_{||}e^{i[(k+\delta k)z-(\omega+\delta\omega)t+\delta_2]}\right\} \tag{H.18}$$

where $\delta\omega$ and δk are the (small) differences in frequencies and wave numbers, and δ_1 and δ_2 are the respective phase angles. We may combine the two terms by using the well-known relationship between the cosine-function and the complex exponentials. The result is

$$E_{||}(z,t) = 2a_{||}\cos[(1/2)(z\delta k - t\delta\omega + \delta')]\Re\left\{e^{i(\overline{k}z-\overline{\omega}t+\overline{\delta})}\right\} \tag{H.19}$$

where $\overline{\omega} = \omega + \frac{1}{2}\delta\omega$, $\overline{k} = k + \frac{1}{2}\delta k$, and $\overline{\delta} = (\delta_1 + \delta_2)/2$. These are the mean angular frequency, the mean wave number, and the mean phase angle, respectively. δ' is the phase angle difference $\delta_1 - \delta_2$. The resultant wave is a plane wave of angular frequency $\overline{\omega}$ and wavelength $2\pi/\overline{k}$ propagating in the z-direction. However, the amplitude of the wave is not constant, but varies with time and position, between the values of $2a_{||}$ and zero (see Fig. AH.1).

This is a mathematical description of the phenomenon of *beats*. The two waves change from being totally in phase (where *constructive interference* occurs) to being totally out of phase (where *destructive interference* occurs). If we set the two frequencies or wave numbers equal, we have two monochromatic plane waves with differing phases, i. e.

$$E_{||}(z,t) = 2a_{||}\cos\left[(1/2)(\delta_1 - \delta_2)\right]\Re\left\{e^{i(kz-\omega t+\overline{\delta})}\right\}. \tag{H.20}$$

When the phases are equal, $\delta_1 = \delta_2$, the amplitude in eqn. H.20 has its maximum value, $2a_{||}$. Again we have constructive interference for in-phase waves. For $\delta_1 - \delta_2 = \pm n\pi$ $(n = 1, 2, \cdots)$, we obtain a zero amplitude for out-of-phase waves, and the destructive interference is complete.

AH.4 Standing Waves

We now consider the superposition of two plane waves travelling in *opposite* directions. This will lead us to the concept of a *standing wave*, a topic of importance to the subject of blackbody radiation. We imagine two oppositely-directed waves of the same frequency, phase and amplitude (the latter we set equal to unity). Again, consider only one component (say the parallel component) of the electric field. The total E-field component is

$$E_{||}(z,t) = \Re\left\{e^{i(kz-\omega t-\pi/2)} + e^{i(-kz-\omega t-\pi/2)}\right\} \quad \text{(H.21)}$$

where we have chosen the phase $\delta = -\pi/2$ for convenience. Using the relationship $\cos kX = (1/2)[\exp(ikX) + \exp(-ikX)]$, we write eqn. H.21 as

$$E_{||}(z,t) = 2\cos(kz + \pi/2)\Re\left\{e^{-i\omega t}\right\} = 2\sin(kz)\cos(\omega t).$$

The result is a wave that neither moves forward or backward. It vanishes at values of z for which $\sin(kz) = 0$, that is, where $kz = n\pi$ $(n = 0, 1, \cdots)$. In between these *nodes*, the disturbance vibrates harmonically with time. The maxima are located at the *anti-nodes*, at $kz = n\pi/2$ $(n = 1, 3, \cdots)$.

For a standing wave located in a finite cavity, the electric field must vanish at the boundaries, say at $z = 0$ and at $z = L$. The nodes will of course correspond with the boundaries, so that $k = n\pi/L$ $(n = 0, 1, \cdots)$. For example, the two lowest-order *wave-modes* are given by

$$E^{(1)}(z,t) = 2\cos(\pi z/L)\cos(\omega t); \qquad E^{(2)}(z,t) = 2\cos(2\pi z/L)\cos(\omega t).$$

The $n = 1$ wave-mode is fixed at the two ends; the $n = 2$ wave-mode is fixed at both ends and in addition is fixed at the center, $z = L/2$. Higher-order wave-modes $E^{(n)}$ have $n + 1$ nodes, etc.

In a three-dimensional cavity (taken to be cubic of sides L for convenience), there are three independent components (actually six, taking into account the perpendicular component). Each has its own wave number, so that

$$k_x = n_x\pi/L; \quad k_y = n_y\pi/L; \quad k_z = n_z\pi/L \qquad (n_x, n_y, n_z = 0, 1, \cdots).$$

In vector notation, we write $\vec{k} = \pi\vec{n}/L$ where $\vec{n}$ is a vector in a three-dimensional pseudo-space with Cartesian components n_x, n_y and n_z.

The above results are applicable to the study of blackbody radiation and is used in the derivation of the *Planck distribution* in §4.3. A radiation field may be thought of as a system of standing waves in a large cavity, or *hohlraum*. The cavity 'walls' are unimportant except for establishing the boundary conditions. In the quantum theory each standing wave may be associated with a *photon*, a particle of light having a quantized energy and momentum given by

$$\begin{aligned}\text{photon energy} &= \mathcal{E}_p = h\nu = \frac{hc}{\lambda} = \frac{h}{2\pi}\omega = \frac{h}{2\pi}c|\vec{k}| \\ \text{photon momentum} &= \mathcal{P}_p = \frac{h\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}|\vec{k}|.\end{aligned}$$

where h is Planck's constant=6.63×10^{-34} $[J \cdot s]$.

In this appendix we found that the linear superposition of electromagnetic fields leads to the phenomena of beating, interference and standing waves. These are all results of *coherent addition* of light waves, and is to be contrasted with the very different situation of *incoherent addition*. It is the latter situation we are mainly concerned with in this book.

AH.5 Polarization

We now consider the way in which the electric field vector of a plane wave varies in space and time. Defining the variable part of the phase factor of eqns. H.9–H.11 as $\phi = kz - \omega t$, we may write the electric field components as

$$E_{||} = a_{||}\cos(\phi + \delta_{||}); \qquad E_{\perp} = a_{\perp}\cos(\phi + \delta_{\perp}). \tag{H.22}$$

We can determine how $\vec{E}$ varies in space by eliminating ϕ. It is easily shown that

$$\left(\frac{E_{||}}{a_{||}}\right)^2 + \left(\frac{E_{\perp}}{a_{\perp}}\right)^2 - 2\frac{E_{||}E_{\perp}}{a_{||}a_{\perp}}\cos\delta = \sin^2\delta \tag{H.23}$$

where $\delta \equiv \delta_{||} - \delta_{\perp}$. This is the equation of an ellipse, which is inscribed into a rectangle whose sides are parallel to the coordinate axes, and whose lengths are $2a_{||}$ and $2a_{\perp}$ (see Fig. AH.2).

At a given point in space, the tip of the electric field vector will therefore trace out an ellipse – the wave is said to be *elliptically polarized.* The properties of the ellipse are determined by three quantities: either $a_{||}, a_{\perp}$ and $\delta = \delta_{||} - \delta_{\perp}$; or by the major and minor axes, a and b, and the angle ψ. The latter is the angle the major axis makes with the

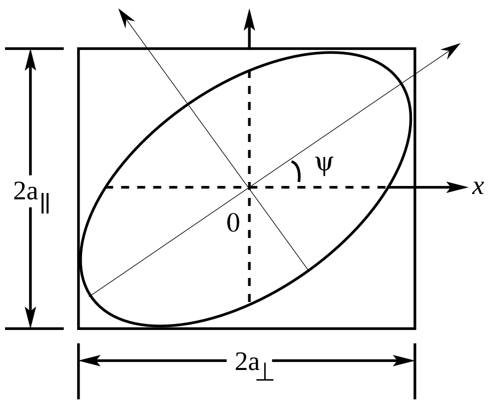

Fig. AH.2. Elliptically polarized wave. The vibrational ellipse for the electric vector. The ellipse is inscribed into a rectangle whose sides are parallel to the coordinate axes whose lengths are $2a_{||}$ and $2a_{\perp}$. The ellipse touches the sides at the points $(\pm a_{||}, \pm a_{\perp}\cos\delta)$ and $(\pm a_{||}\cos\delta, \pm a_{\perp})$.

horizontal (parallel axis) as shown in Fig. AH.2. It may be shown that these quantities are related to the first set by

$$a^2 + b^2 = a_{||}^2 + a_{\perp}^2; \qquad \pm ab = a_{||}a_{\perp}\sin\delta;$$

$$\tan 2\psi = (\tan 2\alpha)\cos\delta; \qquad \tan\alpha = \frac{a_{||}}{a_{\perp}}. \qquad \text{(H.24)}$$

AH.6 Polarization: linear and circular

The special cases of linear and circular polarization occur when the ellipse in eqn. H.23 degenerates into either a straight line or a circle. When the phase difference of the two components is an integral multiple of π, that is when $\delta = \delta_{||} - \delta_{\perp} = m\pi$ for $(m = 0, \pm 1, \pm 2, \cdots)$, eqn. H.23 yields

$$\frac{E_{\perp}}{E_{||}} = (-1)^m \frac{a_{\perp}}{a_{||}}.$$

In this case, $\vec{E}$ is *linearly polarized.* The two components bear a constant ratio to one another. Considering the time-dependent factor ϕ

(see eqn. H.22), we see that the $\vec{E}$-vector oscillates in magnitude (with angular frequency ω) along a straight line, from the value $-a_{||}$ to $+a_{||}$. When the components have equal magnitude, $a_{||} = a_{\perp} = a$, and in addition the phase angles are in quadrature, that is $\delta = \delta_{||} - \delta_{\perp} = m\pi/2$ where $m = (\pm 1, \pm 3, \pm 5, \cdots)$, eqn. H.23 reduces to the equation for a circle, i. e.

$$E_{||}^2 + E_{\perp}^2 = a^2.$$

Additional information on the *Stokes-vector* representation of light, and other advanced topics, is given in Appendix I and in other texts.† In the natural environment light is *partially-polarized* or in some limiting situations, *unpolarized.* Simply stated, the latter means that there is no preference between the parallel- and perpendicular-directions, and no permanent phase relationships exist between these two components. Sunlight, diffuse visible light emanating from an optically-thick cloud cover, and thermal IR emission are important examples of (nearly) unpolarized light. Rayleigh scattering from a clear sky is a counter-example, as the degree of linear polarization of scattered light can be quite high. Despite its importance in some applications, we will ignore polarization on the grounds that we are mainly concerned with the energy flow, rather than the accurate intensity distribution. This is called the *scalar approximation.* Even though caution is advised, it often provides reasonably accurate results even for the directional distribution of radiation. In addition there are ways to estimate the polarization by making first-order corrections to scalar solutions.

AH.7 Problems

H.1. Consider a plane electromagnetic wave propagating in the z-direction through an isotropic, homogeneous medium with conductivity σ and permittivity ϵ. For this geometry Maxwell's equations simplify to

† Plane waves, polarization, and the Stokes parameters are discussed in the following references: Born, M. and E. Wolf, *Principles of Optics*, Chapter 1, MacMillan, New York, 1964. Coulson, K. L.,*Polarization and Intensity of Light in the Atmosphere*, A. Deepak Publ., Hampton, Va., 1988: Kliger, D. S., J. W. Lewis, and C. E. Randall, *Polarized Light in Optics and Spectroscopy*, Academic Press, Boston, 1990. A practical non-mathematical approach is found in Shurcliff, W. A. and S. S. Ballard, *Polarized Light*, Van Nostrand, Princeton, 1964; An influential journal review is Hansen, J. E. and L. D. Travis, Light Scattering in Planetary Atmospheres, *Space Sci. Rev.*, **16**, 527-610, 1974.

$$\frac{\partial^2 E_{||}}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_{||}}{\partial t^2} + \mu\sigma \frac{\partial E_{||}}{\partial t}$$

$$\frac{\partial^2 E_{\perp}}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_{\perp}}{\partial t^2} + \mu\sigma \frac{\partial E_{\perp}}{\partial t}.$$

(a) Show that the electric field strength diminishes along the beam according to the exponential *Extinction Law* (§2.7), that is, the above set of equations has a solution of the form

$$E_{||} = a_{||} \exp\{i[(\omega t - k_o m_r z)] - k_o m_i z\}$$

where

$$k_o = \frac{2\pi}{\lambda_o} = \frac{\omega}{c_o} = \omega\sqrt{\mu_o \epsilon_o}$$

and m_r and m_i are the real and imaginary parts of the complex index of refraction.

(b) Find the expressions for the two quantities, m_r and m_i, and for the speed of light in the medium in terms of the electric and magnetic properties of the medium. Show that the absorption coefficient $\alpha = k_o n_i$ is given by

$$\alpha = \omega \sqrt{\frac{\epsilon\mu}{2}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}\,\right]}.$$

H.2. (a) Show that the real part of the time-average of the product of two complex quantities, A^* and B^* (having the same periodicity) is given by

$$< \Re(A) \cdot \Re(B) >= \frac{1}{2}\Re(AB^*). \qquad \text{(H.25)}$$

(b) Solve for the H-components of the plane wave travelling in a dielectric medium. From these expressions, show that the Poynting vector is given by eqn. H.16.

Appendix I

Representations of Polarized Light

AI.1 Stokes Parameters

In addition to the frequency, three independent quantities are needed to completely specify a time-harmonic electromagnetic plane wave. Since the quantities used in Appendix H are combinations of amplitudes and angles, which have different units, it is more convenient to use quantities having the same dimensions. In 1852 G. G. Stokes introduced his four parameters

$$I = a_{||}^2 + a_{\perp}^2; \qquad Q = a_{||}^2 - a_{\perp}^2; \qquad U = 2a_{||}a_{\perp}\cos\delta; \qquad V = 2a_{||}a_{\perp}\sin\delta. \tag{I.1}$$

Only three of these are independent, since $I^2 = Q^2+U^2+V^2$. We already found that I is the energy carried by the wave. The other parameters are related to the angle ψ $(0 \leq \psi < \pi)$ specifying the orientation of the ellipse (Fig. AI.2), and the *ellipticity* angle, $\chi(-\pi/4 \leq \chi \leq \pi/4)$, which is given by $\tan\chi = \pm b/a$. The relationships are as follows:†

$$Q = I\cos 2\chi\cos 2\psi; \qquad U = I\cos 2\chi\sin 2\psi; \qquad V = I\sin 2\chi. \tag{I.2}$$

AI.2 The Poincaré Sphere

Eqns. I.2 provide a simple geometrical representation of all the different states of polarization: Q, U and V may be regarded as the Cartesian coordinates of a point P on a sphere of radius I, such that 2χ and 2ψ are the spherical coordinates of this point (Fig. AI.2). Every possible state of polarization of a plane wave is represented by a point on this *Poincaré Sphere*, developed by H. Poincaré in 1892. A point in the

† See Born and Wolf, Principles of Optics, pp. 24-31.

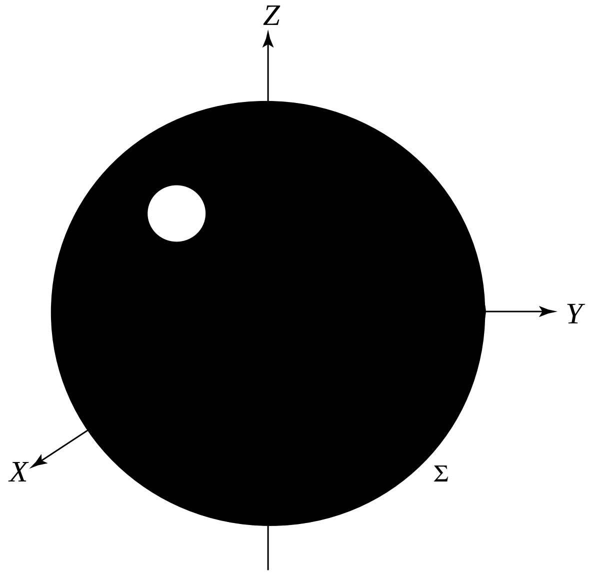

Fig. AI.1. Poincaré's representation of the state of polarization of a monochromatic wave. (The Poincaré sphere).

upper hemisphere (χ positive) represents *right-handed polarization*, that is, when the observer views the wave 'head-on', $\vec{E}$ rotates in a *clockwise* direction. *Left-handed* polarization corresponds to a point in the lower hemisphere (χ negative); when the observer views $\vec{E}$ 'head on', it rotates in a *counter-clockwise* direction. Linear polarization occurs when the phase difference δ is zero, or an integral multiple of π. From eqn. I.1, V is zero, and from eqn. I.2, the z-component of the point P on the Poincaré Sphere is zero. Linear polarization is represented by points in the equatorial plane. For circular polarization, $a_{||} = a_{\perp}$, and $\delta = \pi/2$ or $\delta = -\pi/2$, according to whether the polarization is right- or left-handed. Thus right-handed circular polarization corresponds to the north pole ($Q = U = 0, V = I$), and left-handed circular polarization corresponds to the south pole ($Q = U = 0, V = -I$). *Elliptical polarization* corresponds to a general point on the sphere, other than those in the equatorial plane or at the poles.

The Poincaré Sphere is useful in giving a simple geometrical visualization of the Stokes parameters. It applies only to a light wave which is *perfectly polarized*, an idealization which seldom occurs in nature. We now consider the general situation in which correlation between the two electric field components is not perfect.

AI.3 Partial Polarization and the Incoherency of Natural Light

So far we have assumed that light is a plane wave with constant amplitude and phase difference between the two components. However, a more realistic view is that light is a mixture of plane waves, whose $\vec{E}$-field oscillates over a staggering number of cycles in one second. For example for visible light of $\lambda = 500\ nm$, a wave oscillates at 6×10^{14} cycles in one second. Even a detector with a very short integration time (say, 10^{-4}s) will time-average over many oscillations. The effective Stokes parameters measured by a detector is therefore not the instantaneous values (given by eqn. I.2), but the time-averaged values

$$I = \left\langle a_{||}^2 \right\rangle + \left\langle a_{\perp}^2 \right\rangle ;\ Q = \left\langle a_{||}^2 \right\rangle - \left\langle a_{\perp}^2 \right\rangle ;\ U = \left\langle 2a_{||}a_{\perp} \cos\delta \right\rangle ;\ V = \left\langle 2a_{||}a_{\perp} \sin\delta \right\rangle . \tag{I.3}$$

Generally, a light wave consists of a mixture of waves from different sources, which are statistically uncorrelated over the averaging time of a detector. Suppose we pass the light from such a 'natural' source, e. g. a hot filament, through a filter which passes only a narrow band of frequencies. Even though the frequencies of all the waves are practically equal, the phase angles will differ from one wave to the other. We may visualize the E-components at a point in space as being harmonic in time over immeasurably short time intervals (of the order of $10^{-8} - 10^{-9}\ s$), but 'switching' randomly from one phase angle to another over longer time intervals. If this switching occurs in completely random ways, there will be as many positive phase differences as negative phase differences, or in other words, the time averages of the products $2a_{||}a_{\perp} \cos\delta$ and $2a_{||}a_{\perp} \sin\delta$ will be zero. Similarly, we can visualize the amplitudes being harmonic, and of specific amplitudes over short time intervals, but in a mixture of uncorrelated waves, the average intensities of the two polarization components will be the same, that is, $\langle a_{||}^2 \rangle = \langle a_{\perp}^2 \rangle$. *Thus, for an uncorrelated mixture of plane waves, Q, U, and V all vanish.* This is known as *unpolarized* light. Examples of unpolarized light are direct sunlight, diffuse skylight from an overcast sky, and infrared thermal radiation. However, most scattered light in natural media is partially polarized. It is clear that if some correlation exists between amplitudes or phases, Q, U, and V may be finite, but smaller in value than in the case of a mixture of *coherent* waves. Thus, we see that the difference between coherent and *incoherent* light is the *degree of correlation* between the two E-field components. In this case, the relationship $I^2 = Q^2 + U^2 + V^2$ (valid for fully polarized light; see eqn. I.1) becomes an *inequality*, $I^2 \geq Q^2 + U^2 + V^2$. This property gives us a quantitative measure of the *degree of polarization*, defined as

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \tag{I.4}$$

What is the physical significance of the Stokes parameters? We can relate I, Q, U, and V to a set of ideal measurements, involving a linear polarizer (such as a polaroid filter), and a retardation plate (such as a thin calcite crystal). The polaroid removes the $\vec{E}$-field component of light that passes through in a direction perpendicular to its axis of polarization, and transmits the other component with 100% transmission†. The retardation plate will affect the relative phases of the two components, i. e. it will introduce a relative phase shift, δ. Suppose we have a radiation detector which measures the radiative energy which has passed through a polarizer-retarder combination. It may be shown‡ that the intensity of transmitted light is given by

$$I(\psi, \delta) = (1/2)\left[I' + Q' \cos 2\psi + (U' \cos\delta + V' \sin\delta)\sin 2\psi\right] \tag{I.5}$$

where primed quantities represent the Stokes parameters of the incident light, $\delta = \delta_{||} - \delta_{\perp}$ is the retardation of the $\perp$-component, relative to the $\|$-component, and ψ is the angle of the polarizer axis with the horizontal ($\|$) axis. It is clear from eqn. I.5 that we can use a number of measurements of the incoming beam (varying ψ) to solve for the Stokes parameters of the incident light. If we first consider only a linear polarizer in the beam, so that there is no retardation ($\delta = 0$), and make measurements at $\psi = 0°$, $45°$, $90°$, and $135°$, the first three Stokes parameters may be obtained from these four measurements of $I(\psi, \delta)$:

$$\begin{aligned} I' &= I(0°, 0) + I(90°, 0) \quad &\text{(a)} \\ Q' &= I(0°, 0) - I(90°, 0) \quad &\text{(b)} \\ U' &= I(45°, 0) - I(135°, 0) \quad &\text{(c)}. \end{aligned} \tag{I.6}$$

It is clear from eqn. I.6 that the fourth component V' cannot be measured with a linear polarizer alone: a retarder is needed. Suppose

† Ideally, a polaroid filter would have no effect on that component parallel to the polarization axis, but in all real polaroids, some absorption will take place along this axis also.

‡ The equivalent form of Eqn. I.5 in S. Chandrasekhar, Radiative Transfer, Dover, eqn. 163 (p. 129) has been shown by Hansen and Travis (see endnotes) to have an error in sign. This error arises in the inconsistency between Chandrasekhar's definition of phase difference in his eqn. 154 (p. 28) with the definition of phase difference employed for the Stokes parameters.

we use a polarizer/quarter-wave plate combination. For $\delta = \pi/2$, we get

$$V' = I(45°, \pi/2) - I(135°, \pi/2). \tag{I.7}$$

The physical significance of the Stokes parameters can now be stated in terms of *preferences* as follows: (1) Q gives preference to the $\|$-component over the $\perp$-component; (2) U gives preference to the component making an angle of 45° over that making an angle of 135°; and V gives preference to the 45° component over the 135° component when passed through a polarizer-retarder combination. If unpolarized light were subjected to these measurements, the intensities $I(\psi, \delta)$ would be independent of ψ and δ, so that $Q' = U' = V' = 0$.

If we were to add two polarized light beams together, what is the polarization of the mixture? We found earlier that if we add together two coherent plane waves of the same frequency and amplitude, we obtained an intensity that varies between zero and twice the amplitude of an individual wave. This occurred because of mutual interference, which depended upon the phase angle difference between the two waves. However, if we add together two partially-polarized waves with *no* (time-average) correlation between the phases, *the net result is that the Stokes parameters of the mixture is the sum of the individual Stokes parameters.* This is the most important property of the Stokes parameters. In this book we consider such light mixtures, or in other words, we consider *incoherent light fields.*

Despite our emphasis on incoherent light in this book, it is important to remember that coherent processes are also at work in the natural environment; otherwise we would be deprived of a host of beautiful phenomena, such as rainbows, iridescence, haloes, mirages, etc.† This co-existence of coherent and incoherent light is explained by the notion of *partial coherence*, and the spatial scales over which the various phenomena occur. A natural radiation field is coherent over an inner scale, called the *coherence length, d* (usually $d \approx \lambda$). Thus, light transmitted through a dielectric particle will undergo coherent interaction with its mutual parts, provided the circumference of the particle is of the order of λ. On the other hand, if the particle is much larger than λ, the various beams will behave as if they are refracted and transmitted

† For a lucid description of coherent processes in nature, this classic text should be consulted: M. Minneart, The Nature of Light and Colour in the Open Air, Dover Publications, New York, N. Y., 1954. A new edition of this book was published in 1993 by Springer, with color photos by Pekka Parviainen.

independently‡. In this case, the laws of geometrical optics provide a good description of the overall interaction.

AI.4 The Stokes Vector Representation of Polarized Light

The *Stokes vector* $\vec{I}$ is a four-vector having the four Stokes parameters as its components,

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}. \tag{I.8}$$

In view of the linearity property of light fields, the Stokes vector of a mixture of two incoherent light fields whose Stokes vectors are $\vec{I}_1$ and $\vec{I}_2$ is simply $\vec{I} = \vec{I}_1 + \vec{I}_2$, or

$$\vec{I} = \begin{pmatrix} I_1 \\ Q_1 \\ U_1 \\ V_1 \end{pmatrix} + \begin{pmatrix} I_2 \\ Q_2 \\ U_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} I_1 + I_2 \\ Q_1 + Q_2 \\ U_1 + U_2 \\ V_1 + V_2 \end{pmatrix}. \tag{I.9}$$

The additivity principle also tells us that an unpolarized radiation field can be represented as the sum of two *linearly-polarized* fields which have equal E-field components and have their polarization directions normal to one another. Thus, two linearly-polarized incoherent light fields of equal intensity $(I/2)$ add together to give an unpolarized field:

$$\vec{I} = (I/2)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (I/2)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = I\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{I.10}$$

Note that the first vector in eqn. I.10 has its polarization direction in the $\|$-direction, so that from eqn. I.6, the component in the 90°-direction is zero. The second vector has a zero component in the 0°-direction, so that Q is the negative of that of the first vector.

It is also easy to see that an arbitrarily-polarized light field may be represented by the sum of an unpolarized (u) and a perfectly-polarized

‡ This assumes that the particle is optically homogeneous. If the particle is inhomogeneous, scattering from irregularities causes the internal radiation field to be multiply-scattered, and mutual interference complicates the description. See §3.2 for more discussion.

(p) light field

$$\vec{I} = \vec{I_u} + \vec{I_p} = \begin{pmatrix} I - \sqrt{U^2 + Q^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{U^2 + Q^2 + V^2} \\ Q \\ U \\ V \end{pmatrix} . \qquad \text{(I.11)}$$

In view of the additivity of Stokes parameters it is easy to see why it is possible to represent any arbitrarily-polarized, incoherent radiation field as the linear sum of an unpolarized part I_u and a 100% polarized part, I_p, $I = I_u + I_p$. The degree of polarization is then written $P = I_p/I$, which gives us a more intuitive interpretation of P than provided by eqn. I.4.

For perfectly-polarized light ($\vec{I_u} = 0$), $\vec{I}$ is a vector whose tip lies on the Poincaré sphere. We may visualize partially-polarized light as a vector $\vec{I_p}$, to which is added a 'smeared-out' component of radius $\vec{I_u}$. Over the averaging time period $< t >$, the tip of the vector $\vec{I_u}$ traces out with equal probability all 4π steradians of the Poincaré sphere.

AI.5 The Mueller Matrix

The action of any optical device on an incoherent light beam can be thought of as producing a Stokes vector which is a linear combination of the Stokes components of the light. Formally, we can represent the effect of an optical device in terms of a *Mueller matrix* operation on $\vec{I}$, or in mathematical terms

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{M}\vec{I'} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} . \qquad \text{(I.12)}$$

The input radiation field components are denoted by primes, and the output radiation field components are unprimed. The components M_{ij} may be derived for various types of polarization analyzers, including polaroid filters (or in general *dichroic linear polarizers*) and retarding plates.† We are often concerned with the action of scattering particles on the state of polarization of an incident radiation field. This can also be represented as a linear matrix operator, called the *scattering matrix*, **S**, whose elements depend upon the angle Θ between the incident and scattered wave (the scattering angle), i. e., $S_{ij} = S_{ij}(\Theta)$. In addition

† see Coulson, Polarized Light, pp. 577–584.

S_{ij} depends upon the light-interaction properties of the particles. For the simplest type of scattering, i. e. *Rayleigh scattering*, the scattering matrix is given by

$$\mathbf{S}_{Ray}(\Theta) = \frac{3\sigma}{4\pi} \begin{pmatrix} 1+\cos^2\Theta & \cos^2\Theta - 1 & 0 & 0 \\ \cos^2\Theta - 1 & 1+\cos^2\Theta & 0 & 0 \\ 0 & 0 & 2\cos\Theta & 0 \\ 0 & 0 & 0 & 2\cos\Theta \end{pmatrix} \tag{I.13}$$

where σ is the scattering coefficient, defined in Chapter 3.

The radiation field in atmospheres and oceans can be highly polarized. For example, for clear skies or pure oceans where Rayleigh scattering dominates the radiative transfer, eqn. I.13 shows that for scattering angles near $\Theta = \pi/2$, there is 100% linear polarization for $\Theta = \pi/2$. However, in reality there are slight deviations from this idealized Rayleigh scattering so that the light is about 96% polarized for $\Theta = \pi/2$. (The presence of aerosols reduces this number to no more than 80% in actual cloud-free situations.) Reflection from water or ice surfaces can also lead to high linear polarizations. However, the elliptic component V is always very small, and it is seldom necessary to specify all four Stokes parameters. In fact, since I conveys the information on the energy carried by the field, it is often permissible to ignore the Q and U components as well. This is the principal approximation of this book. We note, however, that although I 'carries the energy', it is sometimes necessary to solve the full *vector* equation (for the Stokes' parameters) to calculate it properly. The *scalar* equation is in many cases adequate as can be confirmed by comparison between *vector* and *scalar* solutions.

AI.6 Problems

I.1. (a) Find the Stokes vector for Rayleigh-scattered light from a small volume element dV having a concentration of n molecules. Use the equation $\vec{I} = n\, dV\, \mathbf{S}_{Ray}\vec{I'}$, where $\mathbf{S}_{Ray}$ is given by eqn. I.13. Assume that the solar intensity is unpolarized and given by

$$\vec{I'} = \begin{pmatrix} F^s\delta(\cos\theta_0 - \cos\theta)\delta(\phi_0 - \phi) \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{I.14}$$

where F^s is the solar flux $[W \cdot m^{-2}]$ and (θ_0, ϕ_0) is the direction of the incoming solar beam.

(b) Describe the state of polarization for Rayleigh scattered light evaluated at the scattering angles $\Theta = \pi/2$ and $\Theta = 0$.

I.2. Devise a number of 'thought experiments' to find the elements of the Mueller matrix for the following optical instruments:

(a) an ideal linear polarizer, e. g. a polaroid filter, with its axis along the horizontal $(\|)$-axis.

(b) the same as (a) but with its axis along the perpendicular $(\perp)$-axis.

Appendix J

Scaling Transformation for Anisotropic Scattering

We will show that the transfer equation is invariant under certain scale changes of the optical depth and the phase function. The so-called $\delta - N$ method, discussed in §6.8.4, turns out to be one such invariant scaling transformation.

We start with the general radiative transfer equation for the total intensity which in slab geometry may be written

$$u\frac{dI(\tau,\hat{\Omega})}{d\tau} = I(\tau,\hat{\Omega}) - \frac{a}{4\pi}\int_{4\pi} dw'\, p(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}') \tag{J.1}$$

where we have ignored the thermal emission term. If we define a kernel

$$G(\hat{\Omega}',\hat{\Omega}) \equiv \frac{1}{4\pi}[-ap(\cos\Theta) + 4\pi\delta(\hat{\Omega}' - \hat{\Omega})] \tag{J.2}$$

then we may rewrite eqn. J.1 as

$$u\frac{dI(\tau,\hat{\Omega})}{d\tau} = \int_{4\pi} G(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}')d\omega'. \tag{J.3}$$

Now, by introducing a new optical depth, $\hat{\tau}$, and a new kernel, $\hat{G}$, through

$$\tau = \beta\hat{\tau} \tag{J.4}$$
$$G = \beta^{-1}\hat{G} \tag{J.5}$$

we find that eqn. J.3 becomes

$$u\frac{dI(\hat{\tau},\hat{\Omega})}{d\hat{\tau}} = \int_{4\pi} \hat{G}(\hat{\Omega}',\hat{\Omega})I(\hat{\tau},\hat{\Omega}')d\omega' \tag{J.6}$$

In view of the definition of G (eqn. J.2) we may rewrite eqn. J.6 as

$$u\frac{dI(\hat{\tau},\hat{\Omega})}{d\hat{\tau}} = I(\hat{\tau},\hat{\Omega}) - \frac{\hat{a}}{4\pi}\int_{4\pi}\hat{p}(\hat{\Omega}',\hat{\Omega})I(\tau,\hat{\Omega}')d\omega' \tag{J.7}$$

where

$$\begin{aligned}\hat{G}(\hat{\Omega}',\hat{\Omega}) &= \frac{1}{4\pi}[-\hat{a}\hat{p}(\cos\Theta) + 4\pi\delta(\hat{\Omega}'-\hat{\Omega})] = \beta G(\hat{\Omega}',\hat{\Omega}) \\ &= \frac{1}{4\pi}[-\beta a p(\cos\Theta) + 4\pi\beta\delta(\hat{\Omega}'-\hat{\Omega})]\end{aligned} \tag{J.8}$$

which implies

$$\hat{a}\hat{p}(\cos\Theta) = [\beta a p(\cos\Theta) + 4\pi(1-\beta)\delta(\hat{\Omega}'-\hat{\Omega})]. \tag{J.9}$$

If we now require the scaled phase function to be normalized to unity as usual, then integration of eqn. J.9 over 4π steradians yields

$$\hat{a} = a\beta + (1-\beta) \tag{J.10}$$

or

$$1-\hat{a} = \beta(1-a). \tag{J.11}$$

This last equation implies that if $a = 1$, then $\hat{a} = 1$, *i. e.* conservative scattering remains conservative under the scaling transformation.

Since expansion of the phase function in Legendre polynomials has been shown to be an extremely useful way of "isolating" the azimuth dependence in slab geometry, we proceed by expanding both phase functions in this manner

$$p(\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)\chi_n P_n(\cos\Theta) \tag{J.12}$$

$$\hat{p}(\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)\hat{\chi}_n P_n(\cos\Theta) \tag{J.13}$$

where $P_n(\cos\Theta)$ is the Legendre polynomial, and the expansion coefficients are defined by eqn. 6.26. The δ-function may also be expanded in Legendre polynomials, i. e.

$$4\pi\delta(\hat{\Omega}'-\hat{\Omega}) = 4\pi\delta(\mu'-\mu)\delta(\phi'-\phi) = 2\delta(1-\cos\Theta) = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\Theta). \tag{J.14}$$

We note that the expansion coefficients in this case are all unity. Substitution of eqns. J.12 and J.14 into eqn. J.6 yields

$$\sum_{n=0}^{\infty}[\hat{a}\hat{\chi}_n - \beta a\chi_n - (1-\beta)](2n+1)P_n(\cos\Theta) = 0 \qquad \text{(J.15)}$$

which implies

$$\hat{a}\hat{\chi}_n = \beta a\chi_n + (1-\beta) \qquad \text{(J.16)}$$

or

$$1 - \hat{a}\hat{\chi}_n = \beta(1 - a\chi_n) \qquad \text{(J.17)}$$

or

$$\hat{\tau}\hat{a}(1-\hat{\chi}_n) = \tau a(1-\chi_n) \qquad \text{(J.18)}$$

where we have used eqns. J.3 and J.9 in the last step. Since $\chi_0 = 1$ eqns. J.16 and J.10 imply $\hat{a}\hat{\chi}_0 = \beta a + 1 - \beta = \hat{a}$ or $\hat{\chi}_0 = 1$. This shows that the expanded *scaled* phase function is correctly normalized as implied by eqn. J.10.

Finally, by defining $h_n = (2n+1)(1 - a\chi_n)$ and using eqn. J.17, we obtain

$$\hat{h}_n = (2n+1)(1-\hat{a}\hat{\chi}_n) = \beta h_n = (\tau/\hat{\tau})h_n \qquad \text{(J.19)}$$

or

$$\hat{\tau}\hat{h}_n = \tau h_n. \qquad \text{(J.20)}$$

The radiative transfer equation is invariant under

scale changes of the optical depth

and phase function which leave invariant the parameter

$$\eta_n \equiv h_n\tau = (2n+1)(1-a\chi_n)\tau \qquad \text{(J.21)}$$

It is clear that $\beta = 1 - af$ in the $\delta - M$ method.

Appendix K

Approximate Solutions of Prototype Radiative Transfer Problems

AK.1 Numerical Implementation of the Discrete Ordinate Method

The solution of the radiative transfer equation described in previous sections has been implemented numerically into a code written in FORTRAN. This code applies to vertically inhomogeneous, nonisothermal, plane-parallel media and it includes all the physical processes discussed previously, namely thermal emission, scattering, absorption, bidirectional reflection and thermal emission at the lower boundary. The medium may be forced at the top boundary by direct (collimated) or diffuse illumination and by internal and boundary sources as well. The coded algorithm is called DISORT (**DIS**crete **O**rdinate **R**adiative **T**ransfer). To make the computer code as clean, robust, and reliable as possible, it was decided to make it highly modularized by constructing many individual subroutines. Each of these subroutines is focused on a particular task, and they are designed to be self-contained, well documented and readable.

The DISORT Fortran-77 code is vailable at:

ftp://climate.gsfc.nasa.gov/pub/wiscombe/Multiple Scatt/

A comprehensive report providing a detailed documentation of the methodology as well as the numerical implementation of the code is also available at the web-site given above.

Appendix L

Spherical Shell Geometry

For solar zenith angles greater than about 80° and twilight situations, we have to take the curvature of the earth into account and solve the radiative transfer equation appropriate for a spherical shell atmosphere.† The geometry is illustrated in Figure AL.1.

In spherical shell geometry, the derivative of the intensity consists of three terms in addition to the one term occurring for slab geometry. These additional terms express the change in the intensity associated

† The treatment of spherical geometry is described in: V. V. Sobolev, *Light Scattering in Planetary Atmospheres* (Transl. by W. M. Irvine), Pergamon, 256 pp., 1975.

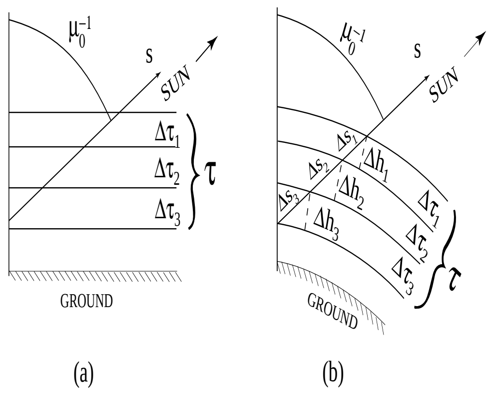

Fig. AL.1. Illustration of plane versus spherical geometry. (a) In plane geometry the slant path is the same for all layers of equal geometrical thickness. (b) In spherical geometry the slant path changes from layer to layer.

with changes in polar angle, azimuthal angle, and solar zenith angle. Hence, for a spherical shell medium illuminated by a direct (collimated) beam of radiation, the appropriate radiative transfer equation for the diffuse intensity may be expressed as (see §6.4)

$$\hat{\Omega} \cdot \nabla I(r, u, \phi, \mu_0) = -k(r)[I(r, u, \phi, \mu_0) - S(r, u, \phi, \mu_0)]. \tag{L.1}$$

Here r is the distance from the center of the planet and k is the extinction coefficient, while u and ϕ are the cosine of the polar angle and the azimuthal angle, respectively. The symbol $\hat{\Omega} \cdot \nabla$ denotes the derivative operator or the 'streaming term' appropriate for this geometry. To arrive at this term we must use spherical geometry. If we map the intensity from a set of global spherical coordinates to a local set with reference to the local zenith direction, then as explained in Appendix O, the streaming term becomes†

$$\begin{aligned}\hat{\Omega} \cdot \nabla \quad \equiv \quad & u\frac{\partial}{\partial r} + \frac{1-u^2}{r}\frac{\partial}{\partial u} \\ & + \frac{1}{r}f(u, \mu_0)\left[\cos(\phi - \phi_0)\frac{\partial}{\partial \mu_0} + \frac{\mu_0}{1-\mu_0^2}\sin(\phi - \phi_0)\frac{\partial}{\partial(\phi - \phi_0)}\right]\end{aligned} \tag{L.2}$$

where the factor f is given by

$$f(u, \mu_0) \equiv \sqrt{1-u^2}\sqrt{1-\mu_0^2}. \tag{L.3}$$

For slab geometry, only the first term contributes. The curvature gives rise to additional terms. Thus, for spherically symmetric geometry, the second term must be added, while the third and fourth terms are required for a spherical shell medium illuminated by direct (collimated) beam radiation. The source function in eqn. L.1 is

$$\begin{aligned}S(r, u, \phi, \mu_0) \quad \equiv \quad & \frac{a(r)}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(r, u', \phi'; u, \phi) I(r, u', \phi', \mu_0) \\ & + \frac{a(r)}{4\pi} p(r, -\mu_0, \phi_0; u, \phi) F^s e^{-\tau Ch(r, \mu_0)}.\end{aligned} \tag{L.4}$$

† The derivation of the 'streaming' term given in Appendix O is taken from: A. Kylling: *Radiation Transport in Cloudy and Aerosol Loaded Atmospheres*, Ph. D. Thesis, University of Alaska, Fairbanks, USA, 1992, and the discussion of the azimuthally-averaged equation from: A. Dahlback and K. Stamnes, *A new spherical model for computing the radiation field available for photolysis and heating at twilight*, Planet. Space Sci., **39**, 671–683, 1991.

The first term in eqn. L.4 is due to multiple scattering and the second term is due to first-order scattering. We have used the diffuse/direct splitting so that eqn. L.1 describes the diffuse radiation field only. We note that for isotropic scattering, the primary scattering 'driving term' becomes isotropic, which implies that the intensity becomes azimuth independent. The argument in the exponential, $Ch(r, \mu_0)$, is the *air-mass factor* or the *Chapman function*: the quantity by which the vertical optical depth must be multiplied to obtain the slant optical path. For a slab geometry, $Ch(r, \mu_0) = 1/\mu_0 = \sec\theta_0$. Other properties of $Ch(r, \mu_0)$ are explored in Problems L.1 and L.2. Hence $\exp[-\tau Ch(r, \mu_0)]$ yields the attenuation of the incident solar radiation of flux F^s (normal to the beam) along the solar beam path.

We find that eqn. L.4 may be written as follows

$$\begin{aligned} S(r, u, \phi, \mu_0) = & \\ & \frac{a(r)}{4\pi} \int_0^{2\pi} d\phi' \int_{-1}^{1} du' \Bigg[\sum_{m=0}^{2N-1} (2 - \delta_{0m}) p^m(\tau, u', u) \cos m(\phi - \phi') \Bigg] I(r, u', \phi') \\ & + \Bigg[\sum_{m=0}^{2N-1} X_0^m(\tau, u) \cos m(\phi - \phi_0) \Bigg] e^{-\tau Ch(r, \mu_0)} \end{aligned} \quad \text{(L.5)}$$

where $p^m(\tau, u', u)$ and $X_0^m(\tau, u)$ are defined by eqns. 6.33 and 6.36.

AL.1 "Isolation" of Azimuth Dependence

The extra derivative terms in eqn. L.2 makes the spherical geometry case more difficult to treat than the corresponding slab problem. In general, we could expand the intensity in a Fourier series containing both *sine* and em cosine terms to account for the appearence of both types of terms in the derivative operator. However, if the effects of sphericity are small, it is useful to treat the second, third, and fourth derivative terms in eqn. L.2 (which are due to the spherical geometry) as a perturbation. Thus, if we ignore these terms, we are left with a plane parallel problem to solve and the derivative terms can be included in an iterative manner by utilizing the plane parallel solutions. Then, since the first term in eqn. L.5 is essentially a Fourier cosine series, and the diffuse intensity described by eqn. L.1 is driven by the second term in eqn. L.5, which contains only *cosine* terms, we may expand the intensity as previously expressed by eqn. 6.34 ignoring *sine* terms.

This is because we have assumed that the third and fourth terms in eqn. L.2, which contain *sine* terms, can be treated as a perturbation and hence can be evaluated in an iterative manner from the plane parallel solutions.

With these assumptions, eqn. L.1 becomes

$$\sum_{m=0}^{2N-1}\left\{u\frac{\partial I^m(r,u,\mu_0)}{\partial r}+\frac{1-\mu_0^2}{r}\frac{\partial I^m}{\partial u}\right.$$
$$\left.+\,k(r)\left[I^m(r,u,\mu_0)-S^m(r,u,\mu_0)\right]\right\}\cos m(\phi_0-\phi)=J(r,u,\phi,\mu_0). \tag{L.6}$$

Here

$$S^m(r,u,\mu_0)\equiv\frac{a(r)}{2}\int_{-1}^{1}p^m(r,u',u)I^m(r,u')du'+X_0^m(r,u)e^{-\tau Ch(r,\mu_0)} \tag{L.7}$$

and

$$J(r,u,\phi,\mu_0)\equiv\frac{1}{r}f(u,\mu_0)\left\{\cos(\phi_0-\phi)\sum_{m=0}^{2N-1}\cos m(\phi_0-\phi)\frac{\partial I^m(r,u,\mu_0)}{\partial\mu_0}\right.$$
$$\left.+\frac{\mu_0}{1-\mu_0^2}\sin(\phi-\phi_0)\sum_{m=0}^{2N-1}m\sin m(\phi-\phi_0)I^m(r,u,\mu_0)\right\}. \tag{L.8}$$

In the following example, we describe how the equations may be solved in a simplified geometry.

Example: Zenith Sky and Mean Intensity – Iterative Approach

If we are interested in only the zenith sky intensity (which is azimuthally independent), then only the $m = 0$ term in eqn. 6.34 contributes. For $m = 0$, the second term in eqn. 6.34 is identically zero. Upon averaging over azimuth the first term becomes proportional to $\partial I^1(r,u,\mu_0)/\partial\mu_0$ and may also be discarded if our interest lies solely in the zenith sky intensity. Thus, the zenith sky intensity is obtained by setting $J(r,u,\mu_0) = 0$ in eqn. L.6 and solving it for $m = 0$ only. Similarly, for isotropic scattering there is no azimuth dependence and the complete solution is again arrived at by setting $J(r,u,\mu_0) = 0$ in eqn. L.6 and solving the equation for $m = 0$ only.

If our interest is in photolysis and heating rates, only the mean intensity is needed. We therefore average eqn. L.6 over azimuth to obtain (see also Appendix O):

$$u\frac{\partial I^0(r,u,\mu_0)}{\partial r}+\frac{1-\mu_0^2}{r}\frac{\partial I^0}{\partial u}+\frac{1}{r}\left[J_1(r,u,\mu_0|I^1)+J_2(r,u,\mu_0|I^1)\right]=-k(r)\left[I^0(r,u,\mu_0)-S^0(r,u,\mu_0)\right]$$

where $S^0(r, u, \mu_0)$ is obtained by setting $m = 0$ in eqn. L.7 and

$$J_1(r, u, \mu_0|I^1) = \frac{1}{2} f(u, \mu_0) \frac{\partial I^1(r, u, \mu_0)}{\partial \mu_0}$$

$$J_2(r, u, \mu_0|I^1) = \frac{1}{2} f(u, \mu_0) \frac{\mu_0}{1 - \mu_0^2} I^1(r, u, \mu_0).$$

We note that J_1 and J_2 depend functionally on the first azimuth-dependent Fourier component of the intensity, I^1, as indicated. Dividing by $-k(r)$, and introducing $d\tau = -k(r)dr$, we obtain

$$u \frac{\partial I(\tau, u)}{\partial \tau} = I(\tau, u) - \frac{a(r)}{2} \int_{-1}^{1} du' p(r, u', u) I(r, u') - S^*(\tau, u)$$

where

$$S^*(\tau, u) \equiv X_0(\tau(r), u) e^{-\tau Ch[\tau, \mu_0]} + \frac{1 - u^2}{kr} \frac{\partial I}{\partial u} + \frac{1}{kr}(J_1 + J_2). \tag{L.9}$$

To simplify the notation, we have dropped the $m = 0$ superscript. If we ignore the three last terms in the expression for $S^*(\tau, u)$, we are left with an equation which is identical to that obtained for plane geometry except that the primary scattering term is evaluated in spherical geometry using the correct path length. We shall refer to this approach, in which the primary scattering driving term is included correctly but the multiple scattering is done in plane geometry, as the 'pseudo-spherical' approximation. Having obtained a 'pseudo-spherical' solution, we may proceed to evaluate the terms we neglected and then solve the equation again including those terms. Repetition of this procedure provides an iteration scheme that is expected to converge if the perturbation terms (i.e., the three last terms on the right side of eqn. L.9) are small compared with the driving term. We shall provide an example of this approach later in the book. Suffice it to say here that this approach has been found to be quite useful for obtaining both the mean intensity and the zenith sky intensity in twilight situations.

In a stratified planetary atmosphere, spherical effects (i. e., the angle derivatives), become important around sunrise and sunset. Thus, the first term in eqn. L.9 is the dominant one and the other terms may be treated as perturbations. It has been shown (by using a perturbation technique to account for the spherical effects) that in a stratified atmosphere, mean intensities may be calculated with sufficient accuracy for zenith angles less than 90° by including only the first term in eqn. L.9, when spherical geometry is used to compute the direct beam attenuation. Then, we may ignore all angle derivatives and simply write the streaming term as

$$\hat{\Omega} \cdot \nabla \cong uk \frac{\partial}{\partial \tau}. \tag{L.10}$$

While this 'pseudo-spherical' approach works adequately for the computation of intensities in the zenith– and nadir-viewing directions, and mean intensities (for zenith angles less than 90°), it may not work for computation of intensities in directions off-zenith (or off-nadir) unless it can be shown that the angle derivative terms are indeed small.

AL.2 Problems

1. The optical depth in a curved atmosphere is required to compute the attenuation of solar irradiance. For an overhead sun, the vertical optical depth between altitude z_0 and the sun is

$$\tau(z_0, \nu) = \int_{z_0}^{\infty} dz k(z, \nu)$$

where $k(z, \nu)$ is the extinction coefficient at frequency ν, and dz is measured along the vertical. For a non-vertical path dz must be replaced by the actual length along the ray path. In slab geometry the actual path length along a ray is simply dz/μ_0 where μ_0 is the cosine of the solar zenith angle. In spherical geometry the situation is somewhat more complex. Then dz must be replaced by the actual ray path through a curved atmosphere.

(a) For solar zenith angles $\theta_0 < 90°$, use geometrical considerations to derive the following expression for the optical depth between level z_0 and the sun in a spherical atmosphere

$$\tau(z_0, \nu, \mu_0) = \int_{z_0}^{\infty} dz \frac{k(z, \nu)}{\sqrt{1 - \left(\frac{R+z_0}{R+z}\right)^2 (1 - \mu_0^2)}} \qquad (\theta_0 < 90°)$$

where R is the radius of the planet and z_0 the distance above the Earth's surface.

(b) Similarly for $\theta_0 > 90°$ show that the following expression applies

$$\begin{aligned} \tau(z_0, \nu, \mu_0) &= 2\int_{z_s}^{\infty} dz k(z, \nu) \left[1 - \left(\frac{R + z_s}{R + z}\right)^2\right]^{-\frac{1}{2}} \\ &\quad - \int_{z_0}^{\infty} dz k(z, \nu) \left[1 - \left(\frac{R + z_0}{R + z}\right)^2 (1 - \mu_0^2)\right]^{-\frac{1}{2}} \end{aligned}$$

where z_s is a screening height below which the atmosphere is essentially opaque to radiation of frequency ν.

For practical computations we may divide the spherical atmosphere into a number of concentric shells. Let Δh_j denote the (vertical) thickness of the shell lying between r_j ($r_j = R + z_j$) and r_{j+1} ($r_{j+1} = r_j - \Delta h_j$)

where z_j is the vertical distance from the surface of the planet to location r_j. (Note that r_1 is at the top of the atmosphere and r_{L+1} is at the bottom of the deepest layer (shell) considered if the atmosphere is divided into L concentric shells.)

(c) Show that approximate expressions for the optical depth that may be used in practical computations are given by

$$\tau(\tau, \nu, \mu_0) = \sum_{j=1}^{p} \Delta\tau_j^v \left(\frac{\Delta S_j}{\Delta h_j}\right) \qquad \theta_0 < 90^\circ$$

$$\tau(\tau, \nu, \mu_0) = \sum_{j=1}^{p} \Delta\tau_j \left(\frac{\Delta S_j}{\Delta h_j}\right) + 2\sum_{j=p+1}^{L-1} \Delta\tau_j \left(\frac{\Delta S_j}{\Delta h_j}\right) + \Delta\tau_L \left(\frac{\Delta S_L}{\Delta h_L}\right) \qquad (\theta_0 > 90^\circ).$$

Here L is the layer in the atmosphere below which attenuation is complete, τ_j is the vertical optical depth of shell j, and

$$\Delta S_j = \sqrt{r_j^2 - r_p^2(1-\mu_0^2)} - \sqrt{r_{j+1}^2 - r_p^2(1-\mu_0^2)}$$

where r_j and r_{j+1} are the distances from the center of the planet to the upper and lower boundary, respectively of layer j, and r_p is the distance from the center to the point at which the optical depth is evaluated.

2. (a) Show that the Chapman function may be written

$$Ch(X, \theta) \equiv \frac{\mathcal{N}(z, \theta)}{n(z)H} = \int_0^\infty dY \exp[-\sqrt{X^2 + 2XY\cos\theta + Y^2} + X].$$

Here $X = R_\oplus/H$, $Y = z/H$, and $\mathcal{N}(z, \theta)$ is the slant column number for a spherically-symmetric exponential atmosphere. (b) Defining $\ln V = -\sqrt{X^2 + 2XY\cos\theta + Y^2} + X$, show that

$$Ch(X, \theta) = \int_0^1 \frac{dV(1 - \ln V/X)}{\sqrt{(1 + \sin\theta - \frac{\ln V}{X})(1 - \sin\theta - \frac{\ln V}{X})}}.$$

(c) Using the relationship

$$\int_0^1 \frac{dV}{\sqrt{\xi^2 - \ln V}} = 2e^{\xi^2} \int_\xi^\infty ds e^{-s^2}$$

show that, on neglecting terms of order X^{-1},

$$Ch(X,\theta) = \sqrt{2X} e^{X\cos^2\theta/2}[1 - erf(\sqrt{X/2}\cos\theta)]$$

where erf is the error function.

(c) Show that, to order X^{-2}, that

$$Ch(X,\theta) = \frac{2\xi e^{\xi^2}}{\cos\theta}[1 - erf(\xi)]$$

where $\xi = \sqrt{X/2}\cot\theta$.

(e) Show that $Ch(X \to \infty, \theta) \to \sec\theta$ for both forms (c) and (d).

Appendix M

Reciprocity for the Bidirectional Reflectance

In this appendix we prove the *Principle of Reciprocity* for the bidirectional reflectance, that is

$$\rho(\nu;\theta',\phi';\theta,\phi) = \rho(\nu;\theta,\phi;\theta',\phi'). \tag{M.1}$$

Referring to Fig. M.1, the proof first determines the exchange of radiative energy between the black elements dA_1 and dA_2 in a *hohlraum* due to reflection of energy by the surface dA_2. The theorem is proven by equating the energy exchange $dE_{\nu 123}$ from 1 to 3 via 2, and the energy

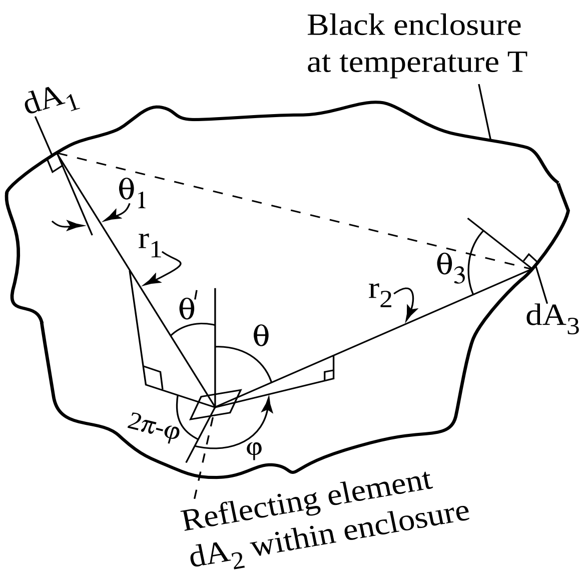

Fig. AM.1. Exchange of radiative energy within a *hohlraum*.

exchange $dE_{\nu 321}$ from 3 to 1 via 2, and equating these two quantities (see Figure AM.1).

The energy exchange $dE_{\nu 123}$ must balance the reciprocal energy exchange $dE_{\nu 321}$ in a TE situation. Otherwise there would be a net heating/cooling of one of the areas at the expense of the other, and this violates the conditions of the *hohlraum*. Let's first consider the exchange from 1 to 3 via 2. The radiative energy reflected by dA_2 and intercepted by dA_3 is given by that 'emitted' into the solid angle subtended by dA_3, $dA_3 \cos\theta_3 / r_2^2$,

$$dE_{\nu 123} = I^{+}_{\nu r2}(\theta', \phi'; \theta, \phi) \cos\theta \frac{dA_3 \cos\theta_3}{r_2^2} d\nu dt. \tag{M.2}$$

The reflected intensity $I^{+}_{\nu r2}$ is related to the intensity $I^{-}_{\nu}(\theta', \phi')$ arriving at dA_2 through eqn. 5.15

$$I^{-}_{\nu r2}(\theta', \phi') = \cos\theta' \rho(\nu; \theta', \phi'; \theta, \phi) I^{-}_{\nu 1}(\theta', \phi') d\omega_{21} \tag{M.3}$$

where

$$d\omega_{21} = \frac{dA_1}{r_1^2} \cos\theta_1 \tag{M.4}$$

is the angle subtended by dA_1 from the point dA_2. Putting these together, we find that the rate of energy exchange per unit frequency from 1 to 3 via 2 is

$$\frac{dE_{\nu 123}}{d\nu dt} = I^{-}_{\nu 1}(\theta', \phi') \rho(\nu; \theta', \phi'; \theta, \phi) \cos\theta' \left[\frac{dA_1}{r_1^2} \cos\theta_1\right] \cos\theta \left[\frac{dA_3}{r_2^2} \cos\theta_3\right]. \tag{M.5}$$

Now consider energy exchange in the reverse direction, 3 to 1 via 2. The rate of energy per unit frequency reflected at dA_2 into the direction of dA_1 is

$$\frac{dE_{\nu 321}}{d\nu dt} = I^{+}_{\nu r2}(\theta, \phi; \theta', \phi') \cos\theta' dA_2 \left[\frac{dA_1}{r_1^2} \cos\theta_1\right]. \tag{M.6}$$

But the reflected intensity at dA_2 is given by

$$I^{+}_{\nu r2}(\theta, \phi; \theta', \phi') = I^{-}_{\nu 3}(\theta, \phi)) \rho(\nu; \theta, \phi; \theta', \phi') \cos\theta d\omega_{23} \tag{M.7}$$

where $d\omega_{23} = dA_3 \cos\theta_3 / r_2^2$ is the solid angle subtended by dA_3 at the point dA_2. Putting these together, we find for the energy exchange rate

$$\frac{dE_{\nu 321}}{d\nu dt} = I^{-}_{\nu 3}(\theta, \phi) \rho(\nu; \theta, \phi; \theta', \phi') \cos\theta' \cos\theta \left[\frac{dA_3}{r_2^2} \cos\theta_3\right] \left[\frac{dA_1}{r_1^2} \cos\theta_1\right]. \tag{M.8}$$

Equating the two rates of energy exchange, we find

$$I_{\nu 1}^{-}(\theta', \phi')\rho(\nu; \theta', \phi'; \theta, \phi) = I_{\nu 3}^{-}(\theta, \phi)\rho(\nu; \theta, \phi; \theta', \phi'). \qquad \text{(M.9)}$$

But in TE, the two intensities are just the Planck function, $I_{\nu 1}^{-} = I_{\nu 3}^{-} = B_\nu$. Therefore

$$\rho(\nu; \theta', \phi'; \theta, \phi) = \rho(\nu; \theta, \phi; \theta', \phi'). \qquad \text{(M.10)}$$

Given the above reciprocity property for the BDRF, we now show that reciprocity also applies to the flux reflectance. Placing the two definitions together, we have

$$\rho(\nu; -\hat{\Omega}', 2\pi) = \int_{+} d\omega \cos\theta \rho(\nu; -\hat{\Omega}', \hat{\Omega}) \qquad \text{(M.11)}$$

$$\rho(\nu; 2\pi, \hat{\Omega}) = \int_{-} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}', \hat{\Omega}). \qquad \text{(M.12)}$$

The first quantity, $\rho(\nu; -\hat{\Omega}', 2\pi)$ is the *directional-hemispherical reflectance*, and the second quantity, $\rho(\nu; 2\pi, \hat{\Omega})$ is the *hemispherical-directional reflectance.* If we evaluate the first of the above equations at $\hat{\Omega}' = \hat{\Omega}$, and place primes on the angular integration variables (realizing that they are dummy variables), we have

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \int_{+} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}, \hat{\Omega}'). \qquad \text{(M.13)}$$

Invoking reciprocity of the BDRF, $\rho(\nu; -\hat{\Omega}, \hat{\Omega}') = \rho(\nu; -\hat{\Omega}', \hat{\Omega})$, we have

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \int_{+} d\omega' \cos\theta' \rho(\nu; -\hat{\Omega}', \hat{\Omega}). \qquad \text{(M.14)}$$

But this is the same expression for the hemispherical-directional reflectance, eqn. M.12. Thus we find the desired reciprocity relationship

$$\rho(\nu; -\hat{\Omega}, 2\pi) = \rho(\nu; 2\pi; \hat{\Omega}). \qquad \text{(M.15)}$$

Appendix N

Isolation of the Azimuth-Dependence

The purpose of this Appendix is to provide a derivation of the azimuthal components of the intensity field. We start with the half-range equations for the diffuse intensity which we write in full-range form for the present purpose

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} \left\{ \mathrm{d}u' p(u',\phi';u,\phi) I(\tau,u',\phi') - \frac{aF^s}{4\pi} p(-\mu_0,\phi_0;u,\phi) e^{-\tau/\mu_0} \right\}. \tag{N.1}$$

Since, as noted in §6.3 the expansion of the phase function in Legendre polynomials is essentially a Fourier cosine series, i.e.

$$p(u',\phi';u,\phi) = \sum_{m=0}^{2N-1} (2-\delta_{0m})\, p^m(u',u)\cos[m(\phi-\phi')], \tag{N.2}$$

where

$$p^m(u',u) = \sum_{l=m}^{2N-1} (2l+1)\chi_l^m \Lambda_l^m(u)\Lambda_l^m(u') \tag{N.3}$$

we expand the intensity likewise

$$I(\tau,u,\phi) = \sum_{m=0}^{2N-1} I^m(\tau,u)\cos[m(\phi_0-\phi)]. \tag{N.4}$$

Substitution of eqns. N.2 and N.4 into the integral term of eqn. N.1 yields

$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') =$$
$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' \left\{\sum_{m=0}^{2N-1} (2-\delta_{0m})\, p^m(u',u)\cos[m(\phi-\phi')]\right\}$$
$$\cdot\left\{\sum_{r=0}^{2N-1} I^r(\tau,u')\cos[r(\phi_0-\phi')]\right\}. \tag{N.5}$$

Focussing on the integration over azimuth we find that for arbitrary m-values only the $r = m$ term contributes. Thus, we obtain $2\pi I^0(\tau,u')$ for $m = 0$, $2\pi I^1(\tau,u') cos(\phi_0 - \phi)$ for $m = 1$, and in general

$$\sum_{m=0}^{2N-1}\int_0^{2\pi} d\phi'(2-\delta_{0m})\sum_{r=0}^{2N-1} I^r(\tau,u')\cos[m(\phi-\phi')]\cos[r(\phi_0-\phi')] =$$
$$2\pi\sum_{m=0}^{2N-1} I^m(\tau,u')\cos[m(\phi_0-\phi)]. \tag{N.6}$$

Therefore eqn. N.5 reduces to

$$\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') =$$
$$\sum_{m=0}^{2N-1}(2-\delta_{0m})\left\{\frac{a}{2}\int_{-1}^{1} du' p^m(u',u) I^m(\tau,u')\right\}\cos[m(\phi_0-\phi)]. \tag{N.7}$$

It is now clear that substitution of eqns. N.2 and N.4 into eqn. N.1 yields the desired result given in Chapter 6, i.e. eqns. 6.33–6.36.

AN.1 Treatment of the Lower Boundary Condition

Since we are dealing with reflection it is natural to use half-range quantities here. The diffuse reflectance at the lower boundary, $\tau = \tau^*$, is written as (see §6.9.4)

$$I^+(\tau^*,\mu,\phi) = \epsilon(\mu)B(T_s) + \frac{\mu_0 F^s}{\pi}\rho_d(-\mu_0,\phi_0;\mu,\phi)e^{-\tau^*/\mu_0}$$
$$+\frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu'\mu'\rho_d(-\mu',\phi';\mu,\phi) I^-(\tau^*,\mu',\phi') \tag{N.8}$$

where ρ_d is the bidirectional reflectance and ϵ is the emittance. First we note that only the $m = 0$ component of the intensity contributes to fluxes, since

$$\begin{aligned} F^{\pm} &= \int_0^{2\pi} d\phi \int_0^1 d\mu\mu I^{\pm}(\tau,\mu,\phi) \\ &= \int_0^{2\pi} d\phi \int_0^1 d\mu\mu \sum_{m=0}^{2N-1} I^{m\pm}(\tau,\mu)\cos[m(\phi-\phi_0)] \\ &= 2\pi \int_0^1 d\mu\mu I^{0\pm}(\tau,\mu). \end{aligned} \tag{N.9}$$

Next we note that Kirchhoff's law states

$$\epsilon(\mu) + \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu'\rho_d(-\mu',\phi';\mu,\phi) = 1 \tag{N.10}$$

suggesting that we should use eqn. N.10 to compute the emittance from the reflectance for consistency. Below we shall start by looking at the simple case of a Lambert reflector before we consider the more general case.

AN.2 Lambertian Surface

A Lambert reflector is defined such that the reflected radiation is isotropic regardless of the directional dependence of the incident radiation. This implies that the bidirectional reflectance is independent of direction, i. e., $\rho_d(-\mu',\phi';\mu,\phi) = \rho_L = constant$. Now, integrating the left side of eqn. N.8, we find that the reflected flux becomes

$$F^+(\tau^*) = \int_0^{2\pi} d\phi \int_0^1 d\mu\mu I^+(\tau^*,\mu,\phi) = \pi I^{0+}(\tau^*) \tag{N.11}$$

since the reflected radiation is isotropic. Integration of the first term on the right side yields $\pi\epsilon B(T_s)$, where we have used Kirchhoff's law yielding $\epsilon(\mu) + \rho_L = 1$, which implies ϵ=constant (independent of μ) in this special case. The second term yields $\rho_L\mu_0 F^s e^{-\tau^*/\mu_0}$, and the third term becomes

$$\int_0^{2\pi} d\phi \int_0^1 \mu d\mu \left[\frac{\rho_L}{\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu' I^-(\tau^*,\mu',\phi')\right] = 2\pi\rho_L \int_0^1 d\mu'\mu' I^{0-}(\tau^*,\mu') \tag{N.12}$$

where $I^{0-}(\tau^*,\mu') = \frac{1}{2\pi}\int_0^{2\pi} I^-(\tau^*,\mu',\phi)d\phi$ is the azimuthally-averaged downward intensity (or the $m = 0$ azimuthal component since we have expressed the intensity in a Fourier cosine series). Thus, for a Lambert reflector we have the following simple boundary condition relating the intensity reflected by the surface to the downward intensity there

$$I^{0+}(\tau^*) = \epsilon B(T_s) + \frac{\mu_0}{\pi} F^s \rho_L e^{-\tau^*/\mu_0} + 2\rho_L \int_0^1 d\mu' \mu' I^{0-}(\tau^*, \mu'). \quad \text{(N.13)}$$

AN.3 Non-Lambertian Surface

We shall assume that the bidirectional reflectance is azimuthally-symmetric so that we may expand it in a Fourier cosine series as

$$\rho_d(-\mu', \phi'; \mu, \phi) = \sum_{m=0}^{2N-1} \rho_d^m(-\mu', \mu) \cos[m(\phi' - \phi)]. \quad \text{(N.14)}$$

In this more general case we find that the third term on the right side of eqn. N.8 becomes

$$\frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu' \mu' \rho_d(-\mu', \phi'; \mu, \phi) I^-(\tau^*, \mu', \phi') = \frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu' \mu' \left\{ \sum_{m=0}^{2N-1} \rho^m(-\mu', \mu) \cos[m(\phi' - \phi)] \cdot \sum_{r=0}^{2N-1} I^{r-}(\tau^*, \mu') \cos[r(\phi_0 - \phi')] \right\}. \quad \text{(N.15)}$$

Since

$$\sum_{m=0}^{2N-1} \int_0^{2\pi} d\phi' \sum_{r=0}^{2N-1} I^{r-}(\tau^*, \mu') \cos[m(\phi' - \phi)] \cos[r(\phi_0 - \phi')] = \pi(1 + \delta_{0m}) \sum_{m=0}^{2N-1} I^{m-}(\tau^*, \mu') \cos[m(\phi_0 - \phi)] \quad \text{(N.16)}$$

we find

$$\frac{1}{\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu' \mu' \rho_d(-\mu', \phi'; \mu, \phi) I^-(\tau^*, \mu', \phi') = \sum_{m=0}^{2N-1} \left\{ 2\int_0^1 d\mu' \mu' \rho^m(-\mu', \mu) I^{m-}(\tau^*, \mu') \right\} \cos[m(\phi_0 - \phi)]. \quad \text{(N.17)}$$

Finally, substitution of eqns. N.4 and N.17 into N.8 yields

$$\sum_{m=0}^{2N-1}\Big\{ I^{m+}(\tau^*,\mu) - \epsilon(\mu)B(T_s)\delta_{0m} - \frac{1}{\pi}F^s\rho_d^m(-\mu_0,\mu)e^{-\tau^*/\mu_0} - (1+\delta_{0m})\int_0^1 d\mu'\mu'\rho_d^m(-\mu',\mu)I^{m-}(\tau^*,\mu')\Big\}\cos[m(\phi_0-\phi)] = 0. \quad \text{(N.18)}$$

Thus, we see that each Fourier component of the intensity must satisfy the boundary condition

$$I^{m+}(\tau^*,\mu) = \epsilon(\mu)B(T_s)\delta_{0m} + \frac{1}{\pi}F^s\rho_d^m(-\mu_0,\mu)e^{-\tau^*/\mu_0} + (1+\delta_{0m})\int_0^1 d\mu'\mu'\rho_d^m(-\mu',\mu)I^{m-}(\tau^*,\mu'). \quad \text{(N.19)}$$

We note that for $m = 0$ and $\rho_d = constant = \rho_L$ we retain the azimuthally-independent case pertinent for a Lambertian surface considered above as we should.

Appendix O

The Streaming term in Spherical Geometry

Since the Earth's atmosphere has the form of a spherical shell, the radiative transfer equation must be cast in a form applicable to spherical geometry. The components of the streaming term $(\hat{\Omega} \cdot \nabla)$ in spherical geometry are

$$\begin{aligned} \hat{\Omega} &= \cos\Phi \sin\Theta \; \mathbf{e}_x + \sin\Phi \sin\Theta \; \mathbf{e}_y + \cos\Theta \; \mathbf{e}_z \qquad (O.1) \\ \nabla &= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_{\Theta_0} \frac{1}{r} \frac{\partial}{\partial \Theta_0} + \mathbf{e}_{\Phi_0} \frac{1}{r \sin\Theta_0} \frac{\partial}{\partial \Phi_0} \end{aligned}$$

where

$$\begin{aligned} \mathbf{e}_r &= \sin\Theta_0 \cos\Phi_0 \; \mathbf{e}_x + \sin\Theta_0 \sin\Phi_0 \; \mathbf{e}_y + \cos\Theta_0 \; \mathbf{e}_z \\ \mathbf{e}_\Theta &= \cos\Theta_0 \cos\Phi_0 \; \mathbf{e}_x + \cos\Theta_0 \sin\Phi_0 \; \mathbf{e}_y - \sin\Theta_0 \; \mathbf{e}_z \\ \mathbf{e}_\Phi &= -\sin\Phi_0 \; \mathbf{e}_x + \cos\Phi_0 \; \mathbf{e}_y \end{aligned}$$

and the angles are defined in Fig AO.1.

Taking the dot product of $\hat{\Omega}$ and ∇ gives

$$\begin{aligned} \hat{\Omega} \cdot \nabla &= [\cos\Theta \cos\Theta_0 + \sin\Theta \sin\Theta_0 \cos(\Phi_0 - \Phi)] \frac{\partial}{\partial r} \\ &\quad - \frac{1}{r} [\cos\Theta \sin\Theta_0 - \sin\Theta \cos\Theta_0 \cos(\Phi_0 - \Phi)] \frac{\partial}{\partial \Theta_0} \\ &\quad - \frac{1}{r} \frac{\sin\Theta}{\sin\Theta_0} \sin(\Phi_0 - \Phi) \frac{\partial}{\partial \Phi_0}. \qquad (O.2) \end{aligned}$$

For practical reasons it is preferable to refer the system of spherical coordinates to the local zenith direction. Thus we want to map the intensity from the set of global coordinates $(r, \Theta_0, \Phi_0, \Theta, \Phi)$ to the local

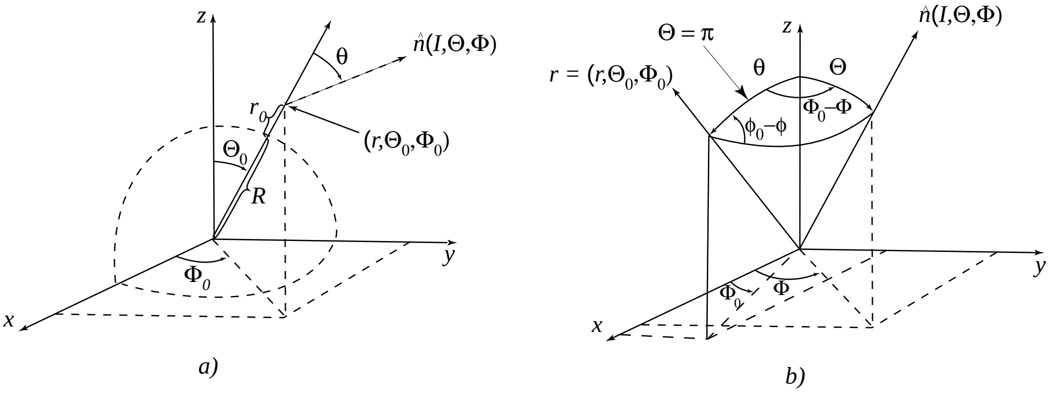

Fig. AO.1. The geometric setting. Note that in panel b the directional vector Ω has been parallel shifted to have its starting point at the surface of the earth.

set $(r, \mu_0, \phi_0, \mu, \phi)$, i.e.†

$$I(r, \Theta_0, \Phi_0, \Theta, \Phi) \quad \Rightarrow \quad I(r, \mu_0, \phi_0, \mu, \phi) \tag{O.3}$$

where

$$\mu \equiv \cos\theta \equiv \mathbf{e}_r \cdot \hat{\Omega} = \cos\Theta\cos\Theta_0 + \sin\Theta\sin\Theta_0\cos(\Phi_0 - \Phi) \tag{O.4}$$

$$\mu_0 \equiv \cos\theta_0 \tag{O.5}$$

and the local polar (θ_0, θ) and azimuthal angles (ϕ_0, ϕ) are defined in Fig. O.1. In view of eqn. O.4 we may rewrite O.2 as

$$\hat{\Omega} \cdot \nabla = \mu \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial \mu}{\partial \Theta_0} \frac{\partial}{\partial \Theta_0} + \frac{1}{r \, \sin^2 \Theta_0} \frac{\partial \mu}{\partial \Phi_0} \frac{\partial}{\partial \Phi_0}. \tag{O.6}$$

Since μ is a function of both Θ_0 and Φ_0

$$\begin{aligned} \frac{\partial}{\partial \Theta_0} &= \frac{\partial}{\partial \theta_0} + \frac{\partial \mu}{\partial \Theta_0} \frac{\partial}{\partial \mu} \\ \frac{\partial}{\partial \Phi_0} &= \frac{\partial \phi_0}{\partial \Phi_0} \frac{\partial}{\partial \phi_0} + \frac{\partial \mu}{\partial \Phi_0} \frac{\partial}{\partial \mu} \end{aligned} \tag{O.7}$$

and eqn. O.6 becomes

$$\hat{\Omega} \cdot \nabla = \mu \frac{\partial}{\partial r} + \frac{1}{r} \left[\left(\frac{\partial \mu}{\partial \Theta_0} \right)^2 + \frac{1}{\sin^2 \theta_0} \left(\frac{\partial \mu}{\partial \Phi_0} \right)^2 \right] \frac{\partial}{\partial \mu} + \frac{1}{r} \frac{\partial \mu}{\partial \Theta_0} \frac{\partial}{\partial \theta_0}$$

† The global coordinates r, Θ_0 and Φ_0 denote a point in $\mathbf{R}^3$, whereas Θ and Φ are the coordinates of a point on the unit sphere $\mathbf{S}^2 = \{x, y : x^2 + y^2 = 1\}$, and similar for the local coordinates. Hence both $I(r, \Theta_0, \Phi_0, \Theta, \Phi)$ and $I(r, \mu_0, \phi_0, \mu, \phi)$ are real–valued functions defined on $\mathbf{R}^3 \times \mathbf{S}^2$.

$$+ \frac{1}{r\ \sin^2\theta_0}\frac{\partial\mu}{\partial\Phi_0}\frac{\partial\phi_0}{\partial\Phi_0}\frac{\partial}{\partial\phi_0}. \tag{O.8}$$

Using eqn. O.4 and some relationships from spherical trigonometry

$$\left[\left(\frac{\partial\mu}{\partial\Theta_0}\right)^2 + \frac{1}{\sin^2\theta_0}\left(\frac{\partial\mu}{\partial\Phi_0}\right)^2\right] = 1-\mu^2 \tag{O.9}$$

$$\frac{\partial\mu}{\partial\Theta_0} = -\cos\Theta\sin\Theta_0 + \sin\Theta\cos\Theta_0\cos(\Phi_0-\Phi) = -\sqrt{1-\mu^2}\cos(\phi_0-\phi) \tag{O.10}$$

$$\frac{\partial\mu}{\partial\Phi_0} = -\sin\Theta\sin\Theta_0\sin(\Phi_0-\Phi) = -\sqrt{1-\mu^2}\sin\theta_0\sin(\phi_0-\phi) \tag{O.11}$$

$$\frac{\partial\phi_0}{\partial\Phi_0} = \frac{\partial(\phi_0-\phi)}{\partial(\Phi_0-\Phi)} = \cos\theta_0\sin(\phi_0-\phi) \tag{O.12}$$

we may finally write the streaming term in spherical geometry referenced to the local zenith direction as

$$\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu} + \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r}\left[\cos(\phi-\phi_0)\frac{\partial}{\partial\mu_0} + \frac{\mu_0}{1-\mu_0^2}\sin(\phi-\phi_0)\frac{\partial}{\partial(\phi-\phi_0)}\right]. \tag{O.13}$$

We note that in plane–parallel geometry only the first term in eqn. O.13 is included. For a spherically symmetric atmosphere the second term must be added. The full expression is, as stated above, valid for an inhomogeneous spherical shell, i.e. a planetary atmosphere.

AO.1 The streaming term pertinent to calculation of mean intensities

Quite generally the intensity may be expanded in a Fourier series

$$I(r,\mu_0,\phi_0,\mu,\phi) = \sum_{m=0}^{\infty}\{I_m^c(r,\mu_0,\mu)\cos m(\phi-\phi_0) + I_m^s(r,\mu_0,\mu)\sin m(\phi-\phi_0)\}. \tag{O.14}$$

Combining eqn. O.13 and eqn. O.14 we find

$$\left\{\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu} + \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r}\cos(\phi-\phi_0)\frac{\partial}{\partial\mu_0}\right\} I(r,\mu_0,\phi_0,\mu,\phi)$$

$$+ \frac{\sqrt{1-\mu^2}\sqrt{1-\mu_0{}^2}}{r}\frac{\mu_0}{1-\mu_0^2}\sin(\phi-\phi_0)$$

$$\cdot \sum_{m=0}^{\infty} \left\{ -m \ I_m^c(r, \mu_0, \mu) \sin m(\phi - \phi_0) + m \ I_m^s(r, \mu_0, \mu) \cos m(\phi - \phi_0) \right\}. \tag{O.15}$$

Since we are interested in the mean intensity

$$\begin{aligned} \overline{I}(r, \theta, \phi) &= \frac{1}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^{\pi} \sin\theta d\theta_0 \ I(r, \theta_0, \phi_0, \phi, \theta) \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\phi_0 \int_{-1}^{1} d\mu_0 \ I(r, \mu_0, \phi_0, \phi, \mu) \end{aligned} \tag{O.16}$$

we average eqn. O.16 over azimuth to get

$$\mu \frac{\partial I_0^c(r, \mu_0, \mu)}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I_0^c(r, \mu_0, \mu)}{\partial \mu} + \frac{1}{2} \frac{\sqrt{1 - \mu^2}\sqrt{1 - \mu_0^2}}{r} \frac{\partial I_1^c(r, \mu_0, \mu)}{\partial \mu_0}$$
$$+ \frac{1}{2} \frac{\sqrt{1 - \mu^2}\sqrt{1 - \mu_0^2}}{r} \frac{\mu_0}{1 - \mu_0^2} I_1^c(r, \mu_0, \mu). \tag{O.17}$$

Note that only the cosine terms 'survived' the averaging over azimuth.

Appendix P

Reciprocity, Duality and Effects of Surface Reflection

The purpose of this Appendix is to provide some details that were omitted in §6.10 regarding the relationship between the reflection and transmission for *unidirectional* (parallel beam or 'solar') and *uniform* (isotropic over the downward hemisphere) illumination of an inhomogeneous slab. The reflectance and transmittance for unidirectional illumination of a slab will be shown to be equivalent to the angular distribution of the azimuthally-averaged reflected and transmitted intensities, respectively, pertaining to uniform illumination of the slab with unit incident intensity. For an *inhomogeneous* slab the transmittance for unidirectional illumination from one side (e.g. the *top*) is equivalent to the angular distribution of the intensity pertaining to illumination from the other side (the *bottom*) of the slab. We will then derive an analytic expression for the intensity reflected from a Lambert surface underlying an inhomogeneeous slab, which in turn is required to derive simple analytic expressions for the reflectance and transmittance of an inhomogeneous slab overlying a partially reflecting surface in terms of the solution pertaining to the same slab overlying a *black* surface.

AP.1 Principle of Reciprocity

If the angular scattering depends only on the scattering angle, i. e. the angle between the direction of incidence and the direction in which the photon is scattered, then the phase function may be written

$$p(\Theta) = p(cos\Theta) = p[uu' + (1-u^2)^{\frac{1}{2}}(1-u'^2)^{\frac{1}{2}}cos(\phi - \phi')] \qquad (P.1)$$

where we have used eqn. 3.22. We see that the phase function satisfies the following relations

$$p(\mu, \phi; \mu', \phi') = p(\mu', \phi'; \mu, \phi) \tag{P.2}$$

$$p(-\mu, \phi; -\mu', \phi') = p(\mu', \phi'; \mu, \phi) \tag{P.3}$$

$$p(\mu, \phi; -\mu', \phi') = p(-\mu', \phi'; \mu, \phi) = p(\mu', \phi'; -\mu, \phi). \tag{P.4}$$

The above relations are usually referred to as Helmholtz' reciprocity principle. They are a consequence of time reversal invariance and they apply to a single scattering event.

AP.2 Homogeneous Slab

For a slab of finite thickness multiple scattering cannot, in general, be neglected. Therefore we do not expect reciprocity to be directly applicable. What is important here is, however, that the above reciprocity relations imply the following reciprocity rules for the reflectance and transmittance of a homogeneous slab of arbitrary (but finite) thickness τ^*

$$\rho(\tau^*; \mu, \phi; \mu_0, \phi_0) = \rho(\tau^*; \mu_0, \phi_0; \mu, \phi) \tag{P.5}$$

$$\mathcal{T}(\tau^*; \mu, \phi; \mu_0, \phi_0) = \mathcal{T}(\tau^*; \mu_0, \phi_0; \mu, \phi). \tag{P.6}$$

The radiation reflected and transmitted by the slab may be expressed as

$$I^+(0, \mu, \mu_0, \phi) = \mu_0 F^s \rho(\tau^*; \mu, \phi; \mu_0, \phi_0) \tag{P.7}$$

$$I^-(\tau^*, \mu, \mu_0, \phi) = \mu_0 F^s \mathcal{T}(\tau^*; \mu, \phi; \mu_0, \phi_0) \tag{P.8}$$

where $\mu_0 F^s$ is the (vertical) flux of the incident 'solar' radiation. Averaging over azimuth, we obtain

$$I^+(0, \mu, \mu_0) = \mu_0 F^s \rho(\tau^*; \mu; \mu_0) \tag{P.9}$$

$$I^-(\tau^*, \mu, \mu_0) = \mu_0 F^s \mathcal{T}(\tau^*; \mu; \mu_0) \tag{P.10}$$

where

$$\rho(\tau^*; \mu, \mu_0) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \rho(\tau^*; \mu, \phi; \mu_0, \phi_0) \tag{P.11}$$

$$\mathcal{T}(\tau^*; \mu, \mu_0) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \mathcal{T}(\tau^*; \mu, \phi; \mu_0, \phi_0). \tag{P.12}$$

AP.2.1 Collimated incidence

The reflectance and transmittance for collimated beam ('solar') incidence are obtained by integration

$$\rho^{beam}(\tau^*,\mu_0) = \frac{2\pi}{\mu_0 F^s}\int_0^1 d\mu\mu I^+(0,\mu,\mu_0)\mu d\mu = 2\pi\int_0^1 d\mu\mu\rho(\tau^*,\mu,\mu_0) \tag{P.13}$$

$$\mathcal{T}^{beam}(\tau^*,\mu_0) = \frac{2\pi}{\mu_0 F^s}\int_0^1 d\mu\mu I^-(\tau^*,\mu,\mu_0) = 2\pi\int_0^1 d\mu\mu\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.14}$$

Another integration yields the spherical albedo and transmittance

$$\bar{\rho}^{beam}(\tau^*) = 2\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu_0) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.15}$$

$$\bar{\mathcal{T}}^{beam}(\tau^*) = 2\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu_0) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.16}$$

The superscript 'beam' is used to remind us that the illumination is collimated, the tilde (˜) sign that it is from below, and the overbar (¯) sign that we are are dealing with a *spherical* quantity.

AP.2.2 Uniform incidence

The angular distributions of the reflected and transmitted intensities for *uniform* illumination with unit incident intensity ($F^s = 1$) are

$$I^{+(uni)}(0,\mu) = 2\pi\int_0^1 d\mu_0 I^+(0,\mu,\mu_0) = 2\pi\int_0^1 d\mu_0\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.17}$$

$$I^{-(uni)}(\tau^*,\mu) = 2\pi\int_0^1 d\mu_0 I^-(\tau^*,\mu,\mu_0) = 2\pi\int_0^1 d\mu_0\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.18}$$

The flux albedo and transmittance ($F^{-(uni)}(0) = \pi$) are given by

$$\frac{F^{+(uni)}(0)}{\pi} = 2\int_0^1 d\mu\mu I^+(0,\mu) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\rho(\tau^*,\mu,\mu_0) \tag{P.19}$$

$$\frac{F^{-(uni)}(\tau^*)}{\pi} = 2\int_0^1 d\mu\mu I^-(\tau^*,\mu) = 4\pi\int_0^1 d\mu\mu\int_0^1 d\mu_0\mathcal{T}(\tau^*,\mu,\mu_0). \tag{P.20}$$

The superscript 'uni' is used to remind us that the illumination is uniform.

AP.2.3 Duality

Since $\rho(\tau^*;\mu,\mu_0) = \rho(\tau^*;\mu_0,\mu)$ and $\mathcal{T}(\tau^*;\mu,\mu_0) = \mathcal{T}(\tau^*;\mu_0,\mu)$, the duality relations given in §6.10 follow by comparing the above expressions for collimated and uniform incidence.

AP.3 Inhomogeneous Slab

The expressions given above pertaining to a homogeneous slab will now be generalized to apply to a vertically inhomogeneous slab. We must distinguish between illumination from the top and the bottom. Thus, considering first illumination from the *top* we find that the same expressions as before (given by eqns. P.13–P.16 and eqns. P.17–P.20 above) apply for unidirectional and uniform illumination, respectively. However, for *unidirectional* and *uniform* illumination from the *bottom* we obtain the following expressions

$$\tilde{\rho}^{beam}(\tau^*,\mu_0) = 2\pi \int_0^1 d\mu\mu\tilde{\rho}(\tau^*,\mu,\mu_0);$$
$$\tilde{I}^{-(uni)}(\tau^*,\mu) = 2\pi \int_0^1 d\mu_0\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0), \qquad \text{(P.21)}$$

$$\tilde{\mathcal{T}}^{beam}(\tau^*,\mu_0) = 2\pi \int_0^1 d\mu\mu\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0);$$
$$\tilde{I}^{+(uni)}(0,\mu) = 2\pi \int_0^1 d\mu_0\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0), \qquad \text{(P.22)}$$

$$\tilde{\bar{\rho}}^{beam}(\tau^*) = 4\pi \int_0^1 d\mu\mu \int_0^1 d\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0);$$
$$\frac{\tilde{F}^{-(uni)}(\tau^*)}{\pi} = 4\pi \int_0^1 d\mu\mu \int_0^1 d\mu_0\tilde{\rho}(\tau^*,\mu,\mu_0) \qquad \text{(P.23)}$$

$$\tilde{\bar{\mathcal{T}}}^{beam}(\tau^*) = 4\pi \int_0^1 d\mu\mu \int_0^1 d\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0);$$
$$\frac{\tilde{F}^{+(uni)}(0)}{\pi} = 4\pi \int_0^1 d\mu\mu \int_0^1 d\mu_0\tilde{\mathcal{T}}(\tau^*,\mu,\mu_0). \qquad \text{(P.24)}$$

The superscript 'beam' is used to remind us that the illumination is collimated, the tilde (˜) sign that it is from below, and the overbar (¯) sign that we are are dealing with a *spherical* quantity.

AP.3.1 Reciprocity and Duality

As noted in §6.10 for an inhomogeneous slab the reflectance and transmittance satisfy the following reciprocity relations

$$\begin{aligned}\rho(\tau^*,\mu,\mu_0) &= \rho(\tau^*,\mu_0,\mu); \qquad \tilde{\rho}(\tau^*,\mu,\mu_0) = \tilde{\rho}(\tau^*,\mu_0,\mu);\\ \mathcal{T}(\tau^*,\mu,\mu_0) &= \tilde{\mathcal{T}}(\tau^*,\mu_0,\mu).\end{aligned} \tag{P.25}$$

A crucial difference between the homogeneous and the *in*homogeneous slab is the reciprocity relating the transmittance due to illumination from one side to the illumination from the other side. Of course, for a homogeneous slab it makes no difference to which side we apply the illumination.

By comparing the expressions pertinent for collimated and uniform incidence and using these reciprocity relations we find that it is now a simple matter to generalize the duality relations for a homogeneous slab to obtain the expressions valid for an inhomogeneous slab provided in §6.10.

AP.4 Derivation of the Reflected Intensity Component I_r

In §6.11 we derived simple analytic expressions for the radiation reflected and transmitted by a slab overlying a partially reflecting (Lambert) surface in terms of the reflected intensity reflected at the lower boundary, I_r. In fact, the quantity

$$\frac{I_r}{\mu_0 F^s} = \frac{\rho_L \mathcal{T}(\mu_0;2\pi)}{\pi(1-\bar{\tilde{\rho}}\rho_L)} = \frac{\rho_L \mathcal{T}(-\hat{\Omega}_0,-2\pi)}{\pi(1-\bar{\tilde{\rho}}\rho_L)}$$

appears in Eqs. 6.79 and 6.80 for the bidirectional reflectance and tranmittance of a slab overlying a Lambertian surface. Below we derive an expression for I_r in terms of the reflectance and transmittance pertinent to an inhomogeneous slab overlying a black (i.e *non-reflecting* surface).

In general, the intensity reflected at the lower boundary, $I^+(\tau^*,\mu,\phi)$, is related to the incident intensity, $I^-(\tau^*,\mu,\phi)$, through

$$I^+(\tau^*,\mu,\phi) = \int_0^{2\pi} d\phi' \int_0^1 d\mu'\mu'\rho(-\mu',\phi';\mu,\phi) I^-(\tau^*;\mu',\phi') \tag{P.26}$$

or by averaging over azimuth

$$I^+(\tau^*,\mu) = 2\pi \int_0^1 d\mu'\mu'\rho(-\mu',\mu) I^-(\tau^*,\mu') \equiv I_r \tag{P.27}$$

where $\rho(-\mu', \phi'; \mu, \phi)$ is the bidirectional reflectance of the surface, and $\rho(-\mu', \mu)$ its azimuthal mean. Here I_r is a constant because we are dealing with a Lambert surface for which $\rho(-\mu', \mu) = \rho_L = constant$.

Next we consider the total reflected intensity $I^+_{tot}(0; \mu, \phi)$, which consists of three separate components: (a) the contribution from the atmosphere assuming a non-reflecting or black lower boundary ($\rho = 0$); (b) the diffusely-transmitted component arising from I_r (see eqn. 5.30); and (c) the directly-transmitted component arising from I_r. In mathematical terms, we write

$$I^+_{tot}(0; \mu, \phi) = I^+(0; \mu, \phi; \rho = 0) + \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' \tilde{\mathcal{T}}_d(\mu', \phi'; \mu, \phi) I_r + I_r e^{-\tau^*/\mu} \tag{P.28}$$

since we have assumed that the reflected intensity is azimuth-independent and given by eqn. P.27. Removing I_r from the integral (it is independent of angle), we combine terms (b) and (c), to obtain the total transmittance

$$I_r \Big[e^{-\tau^*/\mu} + \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' \tilde{\mathcal{T}}_d(\mu'; \mu) \Big]. \tag{P.29}$$

The second term is recognized as the diffuse part of the *hemispherical-directional transmittance* $\tilde{\mathcal{T}}_d(2\pi; \mu)$ pertaining to radiation incident from below. We note the absence of azimuthal dependence. As shown for the flux reflectance in Problem 3.1(b), reciprocity also applies to the *flux transmittance*

$$\tilde{\mathcal{T}}_d(2\pi; \mu) = \tilde{\mathcal{T}}_d(\mu; 2\pi). \tag{P.30}$$

In words, the *hemispherical-directional transmittance* is also the *directional-hemispherical transmittance.* Therefore

$$I^+_{tot}(0; \mu, \phi) = I^+(0; \mu, \phi; \rho = 0) + I_r \tilde{\mathcal{T}}(\mu; 2\pi) \tag{P.31}$$

where $\tilde{\mathcal{T}}(\mu; 2\pi) = e^{-\tau^*/\mu} + \tilde{\mathcal{T}}_d(\mu; 2\pi)$, or the total transmittance is the sum of the beam and diffuse transmittances. We note that the remaining ϕ-dependence of the total intensity is due to the first term, and is traceable to a ϕ-dependence of the collimated beam illumination. The extra term in the above equation (arising from the boundary) is azimuthally independent by assumption (Lambert reflector).

The first term may be expressed in terms of the incident radiation field (assumed to be a collimated solar beam) and the atmospheric reflectance as $\mu_0 F^s \rho(-\mu_0, \phi_0; \mu, \phi)$. Therefore

$$I^+_{tot}(0; \mu, \phi) = \mu_0 F^s \rho(-\mu_0, \phi_0; \mu, \phi) + I_r \tilde{\mathcal{T}}(\mu; 2\pi). \tag{P.32}$$

Proceeding in a similar manner we find that the transmitted intensity can be expressed as

$$I_{tot}^{-}(\tau^*;\mu,\phi) = \mu_o F^s \mathcal{T}(-\mu_0,\phi_0;\mu,\phi) + I_r \tilde{\rho}(\mu;2\pi). \quad \text{(P.33)}$$

Here the first term is the diffusely transmitted intensity, while the second term stems from radiation reflected first from the surface and then from the atmosphere above.

It remains to determine I_r. Setting the reflected flux πI_r equal to a constant, ρ_L, times the downward flux at τ^*, we have

$$\pi I_r = \rho_L \left[\mu_0 F^s e^{-\tau^*/\mu_0} + \mu_0 F^s \mathcal{T}_d(\mu_0;2\pi) + \pi I_r \tilde{\tilde{\rho}}\right]. \quad \text{(P.34)}$$

The first term on the left side is the directly-transmitted solar flux, the second term is the diffusely-transmitted component for a completely black surface, and the third term is the (downward) reflected component due to the upward reflection from the Lambert surface followed by downward reflection by the atmosphere. We recognize $\tilde{\tilde{\rho}}$ as the *spherical albedo* pertaining to illumination from below. Solving the above for I_r we obtain

$$I_r = \frac{\mu_0 F^s \rho_L \left[e^{-\tau^*/\mu_0} + \mathcal{T}_d(\mu_0;2\pi)\right]}{\pi(1-\tilde{\tilde{\rho}}\rho_L)} = \frac{\mu_0 F^s \rho_L \mathcal{T}(\mu_0;2\pi)}{\pi(1-\tilde{\tilde{\rho}}\rho_L)} \quad \text{(P.35)}$$

where we have once again combined the sum of the direct and diffuse transmittances into a total transmittance.

Appendix Q

Removal of Overflow Problems in the Intensity Formulas

We start looking at the homogeneous part of eqn. 8.67 for the upward intensity, which we rewrite as (using $k_{-jp} = -k_{jp}$)

$$I_p^+(\tau,\mu) = \sum_{n=p}^{L}\sum_{j=1}^{N}\{C_{-jn}\frac{\tilde{g}_{-jn}(+\mu)}{1-k_{jn}\mu}e^{-[-k_{jn}\tau_{n-1}+(\tau_{n-1}-\tau)/\mu]} - e^{-[-k_{jn}\tau_n+(\tau_n-\tau)/\mu]} + C_{+jn}\frac{\tilde{g}_{+jn}(+\mu)}{1+k_{jn}\mu}e^{-[k_{jn}\tau_n+(\tau-\tau_n)/\mu]} - e^{-[k_{jn}\tau_{n-1}+(\tau-\tau_{n-1})/\mu]}\}. \quad \text{(Q.1)}$$

Introducing eqns. 8.57 into eqn. 8.67, we find

$$I_p^+(\tau,\mu) = \sum_{n=p}^{L}\sum_{j=1}^{N}\{C'_{-jn}\frac{\tilde{g}_{-jn}(+\mu)}{1-k_{jn}\mu}E'_{-jn}(\tau,+\mu) + C'_{+jn}\frac{\tilde{g}_{+jn}(+\mu)}{1+k_{jn}\mu}E'_{+jn}(\tau,+\mu)\} \quad \text{(Q.2)}$$

where

$$E'_{-jn}(\tau,+\mu) = \exp\left[-(k_{jn}\Delta\tau_n + \delta\tau/\mu)\right] - \exp\left[-(\tau_n-\tau)/\mu\right] \quad \text{(Q.3)}$$

with

$$\begin{cases} \Delta\tau_n = \tau_n - \tau_{n-1}, & \delta\tau = \tau_{n-1} - \tau \quad \text{for } n < p \\ \Delta\tau_p = \tau_p - \tau, & \delta\tau = 0 \quad \text{for } n = p \end{cases}, \quad \text{(Q.4)}$$

$$E'_{+jn}(\tau,+\mu) = \exp\left[-(\tau_{n-1}-\tau)/\mu\right] - \exp\left\{-\left[k_{jn}(\tau_n-\tau_{n-1}) + (\tau_n-\tau)/\mu\right]\right\} \quad \text{(Q.5)}$$

for $n > p$ and

$$E'_{+jp}(\tau,+\mu) = \exp[-k_{jp}(\tau-\tau_{p-1})] - \exp\left\{-[k_{jp}(\tau_p-\tau_{p-1}) + (\tau_p-\tau)/\mu]\right\}. \quad \text{(Q.6)}$$

Since $k_{jn} > 0$ for $n = p+1, p+2, \ldots L$ and $\tau_L > \cdots > \tau_{n=p+1} > \tau_{n-1=p} > \tau$ and also $k_{jp} > 0$ and $\tau_{p-1} < \tau < \tau_p$, all the exponentials in eqns. Q.3–Q.6 have negative arguments as they should.

Similarly, by introducing eqns. 8.57 into the homogeneous part of eqn 8.68, we find that the expression for the downward intensity becomes

$$\begin{aligned} I_p^-(\tau,\mu) &= \sum_{n=1}^{p}\sum_{j=1}^{N}\{C'_{-jn}\frac{\tilde{g}_{-jn}(-\mu)}{1+k_{jn}\mu}E'_{-jn}(\tau,-\mu) \\ &\quad +C'_{+jn}\frac{\tilde{g}_{+jn}(-\mu)}{1-k_{jn}\mu}E'_{+jn}(\tau,-\mu)\} \end{aligned} \tag{Q.7}$$

where

$$E'_{+jn}(\tau,-\mu) = \exp[-(k_{jn}\Delta\tau_n + \delta\tau/\mu)] - \exp[-(\tau-\tau_{n-1})/\mu] \tag{Q.8}$$

with

$$\begin{cases} \Delta\tau_n = \tau_n - \tau_{n-1}, & \delta\tau = \tau - \tau_n \quad \text{for } n < p \\ \Delta\tau_p = \tau_p - \tau_{p-1}, & \delta\tau = 0 \quad \text{for } n = p \end{cases}, \tag{Q.9}$$

$$E'_{-jn}(\tau,-\mu) = \exp[-(\tau-\tau_n)/\mu] - \exp\{-[k_{jn}(\tau_n-\tau_{n-1}) + (\tau-\tau_{n-1})/\mu]\} \tag{Q.10}$$

for $n < p$ and

$$E'_{-jp}(\tau,-\mu) = \exp[-k_{jp}(\tau_p-\tau)] - \exp\{-[k_{jp}(\tau_p-\tau_{p-1}) + (\tau-\tau_{p-1})/\mu]\}. \tag{Q.11}$$

Again, we see that all exponentials involved in the scaled solutions have negative arguments since $k_{jn} > 0$ and $\tau > \tau_n > \tau_{n-1}$ for $n = 1, 2, \ldots, p-1$, and also $k_{jp} > 0$ and $\tau_{p-1} < \tau < \tau_p$. This ensures that fatal overflow errors are avoided in the computations.

Appendix R

Integration of the Planck Function across an Arbitrary Spectral Interval

Our problem in integrating the radiative transfer equation across a spectral interval $\Delta\nu$ is that we must approximate integrals of the form

$$\int_{\Delta\nu} B_\nu \Psi_\nu d\nu \tag{R.1}$$

where Ψ_v is the product of the radiative intensity and possibly other frequency-dependent factors. The radiative intensity is given by the Planck function (§4.4):

$$I_\nu^{BB} = B_\nu(T) \equiv \frac{m_r^2}{c^2} \frac{2h\nu^3}{\left(e^{h\nu/k_BT} - 1\right)}.$$

A sound procedure which conserves energy and is rigorously correct in the limit of zero or infinite absorption, is to approximate this integral as

$$\int_{\Delta\nu} B_v \Psi_\nu d\nu = \frac{\xi_{\Delta\nu}}{\Delta\nu} \int_{\Delta\nu} \Psi_\nu d\nu \tag{R.2}$$

where

$$\xi_{\Delta\nu} = \int_{\nu_1}^{\nu_2} B_\nu d\nu = \xi_{\nu_2} - \xi_{\nu_1} \qquad \text{and} \qquad \xi_\nu = \int_0^\nu B_{\nu'} d\nu'. \tag{R.3}$$

Applying the mean value theorem by pulling B_ν through the integral sign and evaluating it at some wavenumber inside $\Delta\nu$ is a much inferior procedure that neither conserves energy nor gives the correct answer in limiting cases (such as when there is no radiatively active medium above a black surface).

A well-documented numerical procedure for evaluating the Planck function is provided in the DISORT report available at the following web-site:

ftp://climate.gsfc.nasa.gov/pub/wiscombe/Multiple Scatt/

A FORTRAN-77 subroutine for computing the Planck function is also available at this web-site as part of the DISORT code.

Appendix S

Computation of the Normalized Associated Legendre Polynomials

We follow the scheme introduecd by Dave and Armstrong† to compute Λ_l^m, the normalized associated Legendre polynomials.

The only stable recurrence for the Λ_l^m is the one involving its subscript. We take it from Dave/Armstrong Eq. (10) but set $l = k + 1$ and $m = n - 1$ to conform to usual notation

$$\Lambda_l^m(u) = \frac{(2l-1)u\Lambda_{l-1}^m(u) - \sqrt{(l+m-1)(l-m-1)}\Lambda_{l-2}^m(u)}{\sqrt{(l-m)(l+m)}} \qquad (l = m+2, \ldots). \tag{S.1}$$

To initialize this recurrence, we need Λ_m^m and Λ_{m+1}^m. Dave and Armstrong express them as multi-step recurrences, but it is possible to derive simpler expressions requiring only a single recursive step. These derivations are provided in the DISORT report that is available at the follwing web-site:

ftp://climate.gsfc.nasa.gov/pub/wiscombe/Multiple Scatt/

A FORTRAN-77 subroutine for computing the Λ_l^m is also available at this web-site as part of the DISORT code.

† Dave, J. V. and B. H. Armstrong, *Computation of High-Order Associated Legendre Polynomials*, J. Quant. Spectros. Radiat. Transfer, 10, 557–562, 1970.