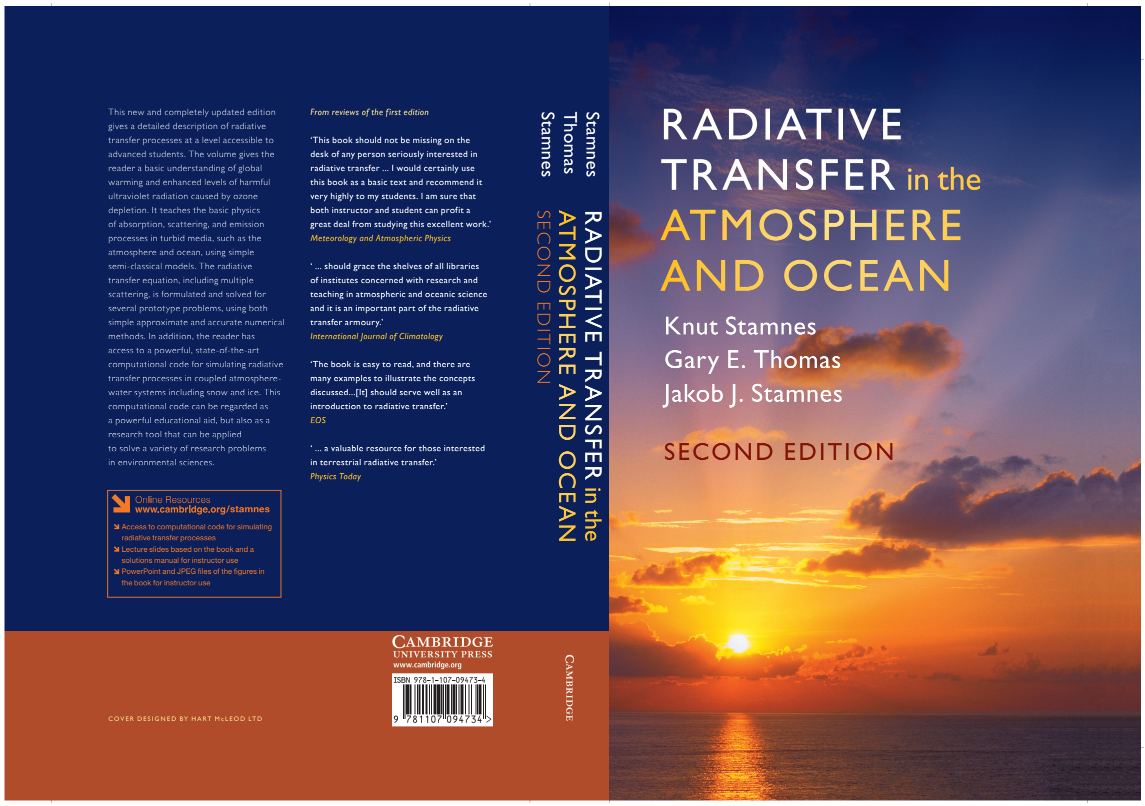
Lecture *Notes:*
Formulation of Radiative Transfer Problems – II



Based on Chapter 6 in K. Stamnes, G. E. Thomas, and J. J. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, 2017.

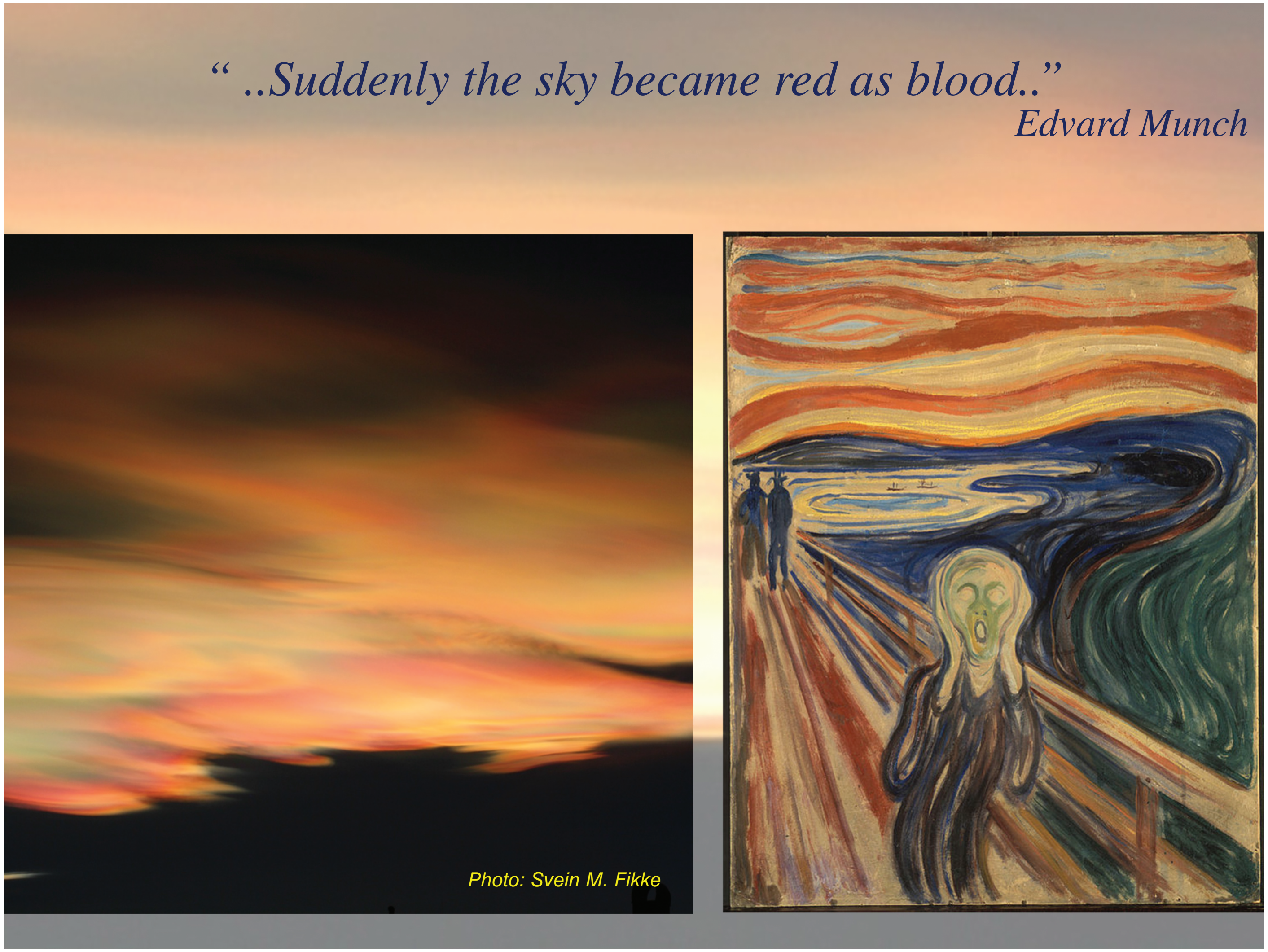


Figure 3. Collage showing details of mother-of-pearl clouds together with The Scream (1910 version).

Figure 6. Pictures of mother-of-pearl clouds presented in Carl Störmer (1948): 'Mother-of-pearl clouds'. (a) 16 February 1946 (Source: Ellen Störmer). (b) 30 January 1944 (Source: Anders Nygaard). (c) 30 January 1944 (Source: Carl Störmer). (d) Same cloud as the one above (i.e. (b)) (Source: Carl Störmer). Störmer wrote: […] photographs in natural colours are given for the first time […]

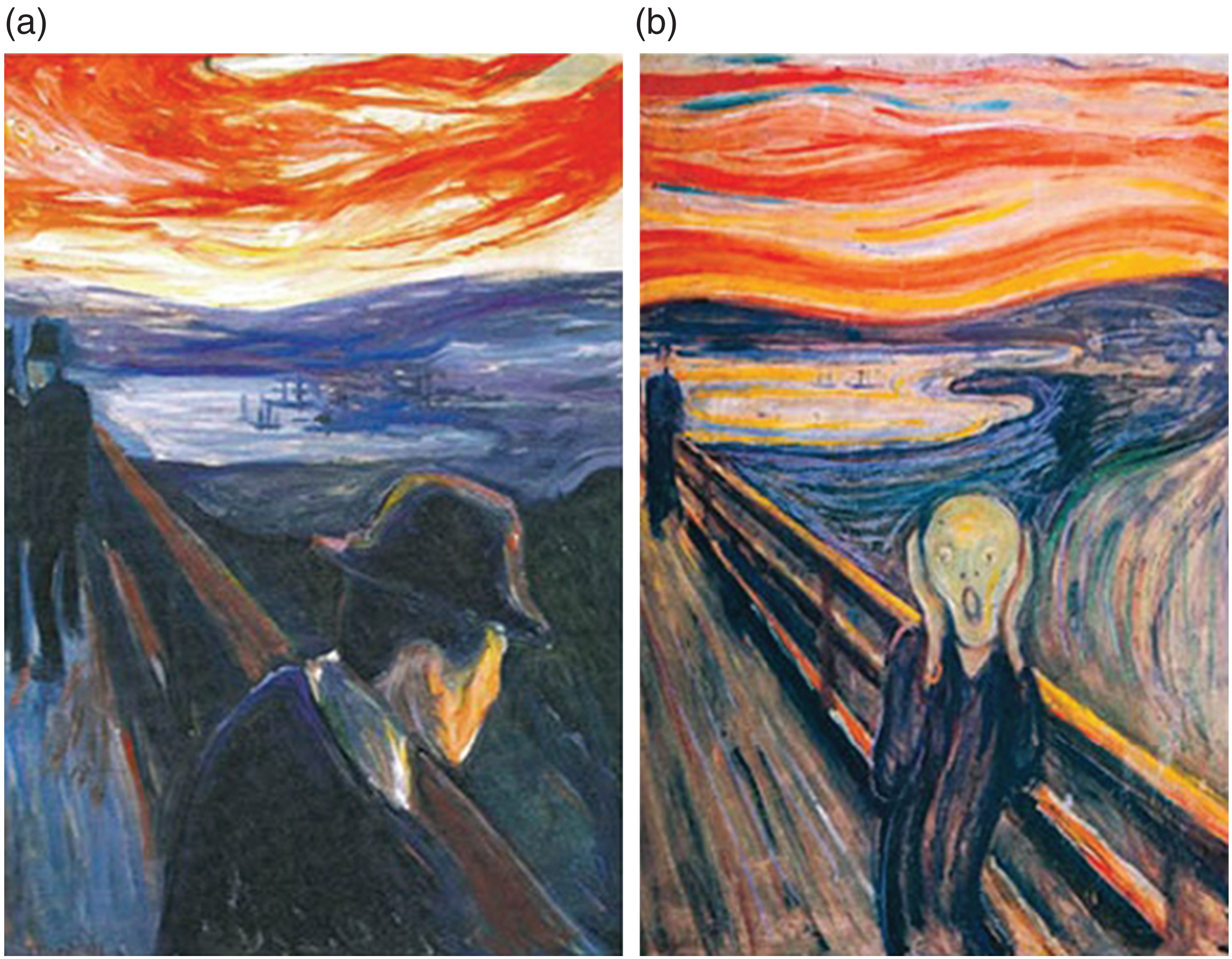


Figure 7. (a) Edvard Munch 'Sick mood by sunset. Despair', 1892. (b) 'The Scream', 1893. Blaafarveverket (2013).

Basic Equations and Definitions

van de Hulst (1957): Let us consider a beam of light with a certain frequency and traveling in one direction. We choose a *plane of reference* through the direction of propagation. By **r** we shall denote the unit vector along the normal of this plane (sense arbitrary) and by **l** the unit vector in this plane and perpendicular to the direction of propagation. The sense is such that $\mathbf{r} \times \mathbf{l}$ is in the direction of propagation. The two letters stand for the last letters of the words perpendicular and parallel.

We may use the Stokes vector representation (superscript ^{T} denotes transpose):

$$\mathbf{I} = [I_\ell, I_r, U, V]^T = [I_\|, I_\perp, U, V]^T$$

In terms of the complex transverse electric field components of the radiation field $E_\ell = |E_\ell| \mathrm{e}^{-i\epsilon_1}$ and $E_r = |E_r| \mathrm{e}^{-i\epsilon_2}$, these Stokes vector components are ($\delta = \epsilon_1 - \epsilon_2$):

$$\begin{aligned} I_\ell &= E_\ell E_\ell^* \\ I_r &= E_r E_r^* \\ U &= 2|E_\ell||E_r| \cos\delta \\ V &= 2|E_\ell||E_r| \sin\delta. \end{aligned} \tag{1}$$

Theory of Vector (Polarized) Radiative Transfer (2)

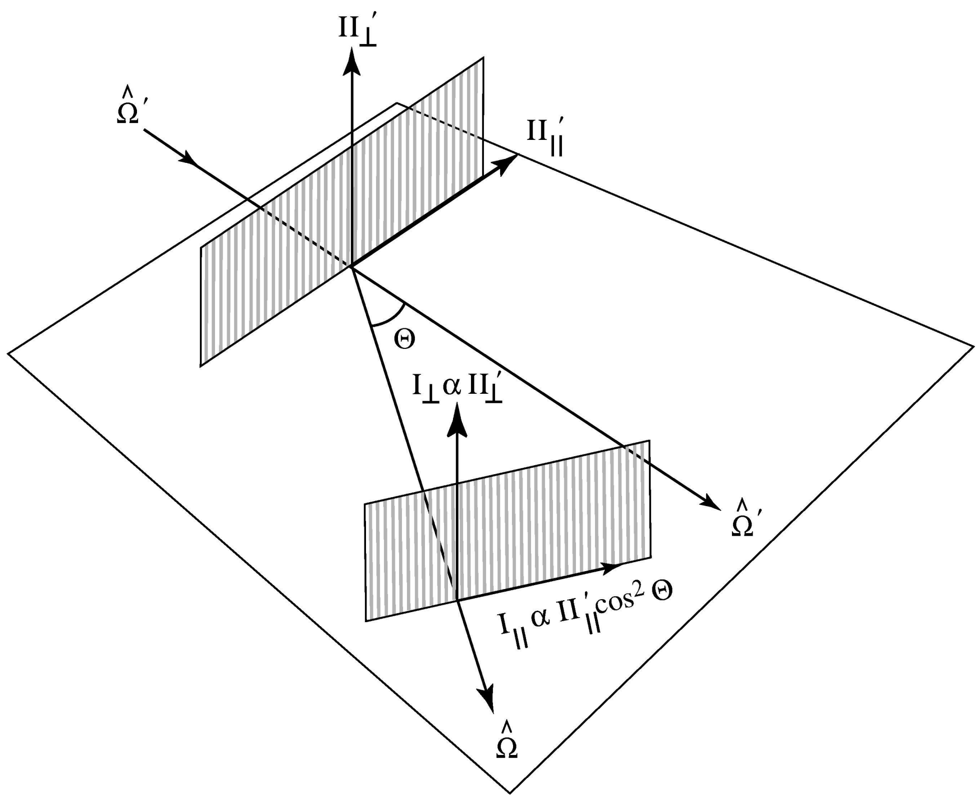


Figure 1: **Illustration of the two transverse components of Rayleigh-scattered light. $\hat{\Omega}'$ and $\hat{\Omega}$ are the incident and scattered propagation vectors, respectively. $\Pi'_\perp$ and $\Pi'_\parallel$ are the induced dipole moments for incident electric fields that are linearly polarized in the directions perpendicular to, and parallel with, the scattering plane (shown as the white rectangle), respectively. $I_\perp$ and $I_\parallel$ are the corresponding scattered radiances in direction $\hat{\Omega}$ associated with the induced dipoles. The plane defined by $\Pi'_\perp$ and $\Pi'_\parallel$ as well as by $I_\perp$ and $I_\parallel$ (both shown as shaded) are normal to the scattering plane.**

The connection between this Stokes vector representation, $\mathbf{I} = [I_\ell, I_r, U, V]^T$, and the more commonly used representation $\mathbf{I}_{\rm S} = [I, Q, U, V]^T$ is simply given by:

$$\mathbf{I}_{\rm S} = \mathbf{D}\mathbf{I} \tag{2}$$

$$\mathbf{D} \equiv \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

The scattered transverse electric field $[E_\ell, E_r]^T$ can be obtained in terms of the incident field $[E_{\ell 0}, E_{r0}]^T$ by a linear transformation:

$$\begin{pmatrix} E_\ell \\ E_r \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} E_{\ell 0} \\ E_{r0} \end{pmatrix}$$

where $\mathbf{A}$ is a 2×2 matrix $\longrightarrow$ the amplitude scattering matrix.

Theory of Vector (Polarized) Radiative Transfer (4)

The corresponding linear transformation connecting:

- the incident and scattered Stokes vectors in the *scattering plane* is called the *Mueller matrix* (for a single scattering event).

For scattering by a small volume containing an ensemble of particles:

- the *ensemble-averaged Mueller matrix* is referred to as the *Stokes scattering matrix* $\mathbf{F}_{\mathrm{S}}(\Theta)$. Here Θ is the scattering angle given by ($u = \cos\theta$):

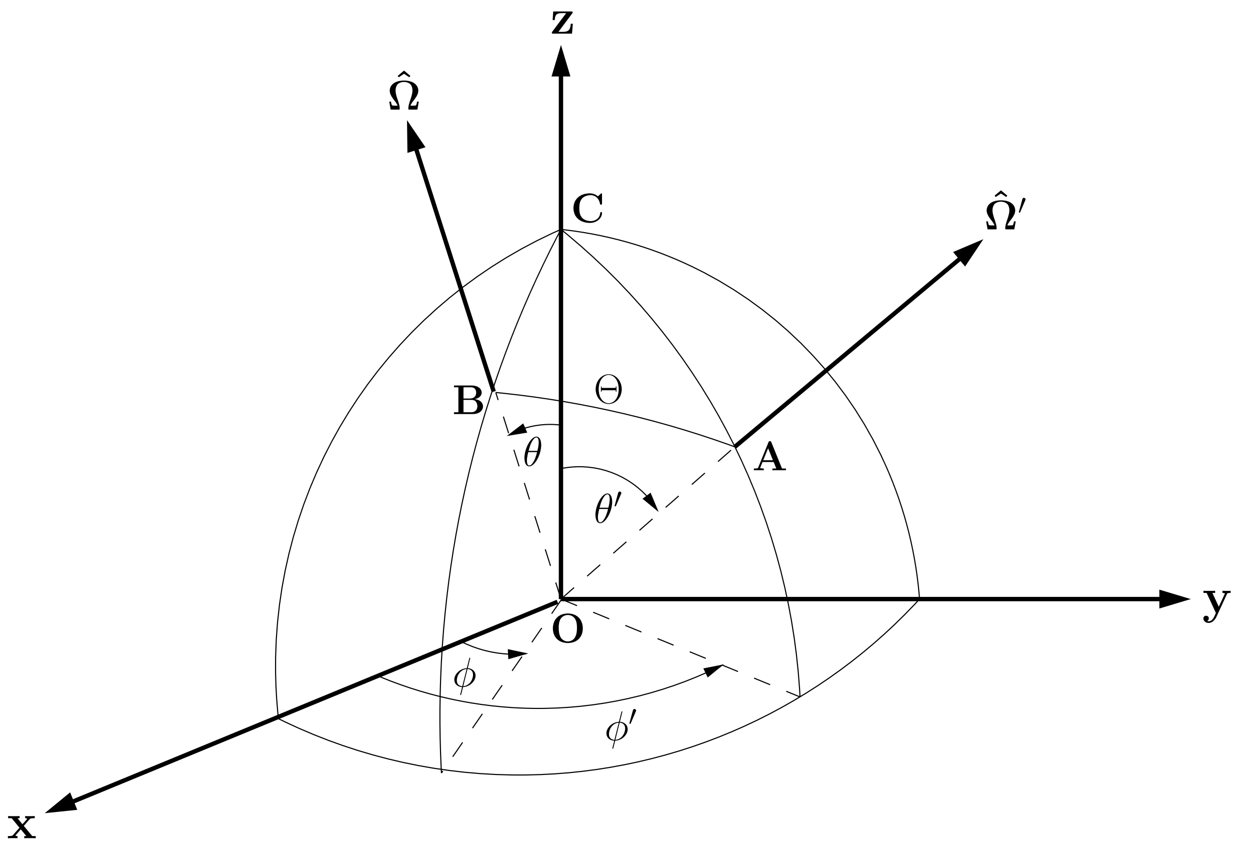
$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi) = uu' + \sqrt{1-u^2}\sqrt{1-u'^2}\cos(\phi' - \phi)$$

where (θ', ϕ') are the polar and azimuthal angles prior to scattering, and (θ, ϕ) those after scattering. Finally, when *transforming from the scattering plane to local meridian planes of reference:*

- the corresponding matrix is referred to as the *scattering phase matrix* $\mathbf{M}_{\mathrm{S}}(u', \phi', u, \phi)$, related to $\mathbf{F}_{\mathrm{S}}(\Theta)$ through:

$$\mathbf{M}_{\mathrm{S}}(u', \phi', u, \phi) = \mathbf{L}(\pi - i_2)\mathbf{F}_{\mathrm{S}}(\Theta)\mathbf{L}(-i_1) \quad (4)$$

where $\mathbf{L}$ is a rotation matrix used to rotate the reference planes.



Coordinate system for scattering by a volume element at **O**. The points **C**, **A**, and **B** are located on the unit sphere. The incident light beam with Stokes vector $\mathbf{I}_{\mathrm{S}}^{\mathrm{inc}}$ is in direction $\mathbf{AO}(\theta', \phi')$ with unit vector $\hat{\mathbf{\Omega}}'$, and the scattered Stokes vector $\mathbf{I}_{\mathrm{S}}^{\mathrm{sca}}$ is in direction $\mathbf{OB}(\theta, \phi)$ with unit vector $\hat{\mathbf{\Omega}}$.

Stokes scattering matrix – *spherical* particles

For *homogeneous spherical particles* the amplitude scattering matrix $\mathbf{A}$ is *diagonal*, and the Stokes scattering matrix in the Stokes vector representation (Eq. 2) is of the following form:

$$\mathbf{F}_{\mathrm{S}}(\Theta) = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_3 \end{pmatrix} \tag{5}$$

where each of the *four independent components* $a_1(\Theta), a_3(\Theta), b_1(\Theta), b_2(\Theta)$ is a function of the scattering angle Θ, given by:

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\phi. \tag{6}$$

$\Delta\phi = \phi - \phi'$ is the difference in azimuth between the direction of incidence (θ', ϕ') and scattering (θ, ϕ).

Generalization to *nonspherical* particles

To include scattering by nonspherical particles, we may adopt a Stokes scattering matrix of the form:

$$\mathbf{F}_{\mathrm{S}}(\Theta) = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{pmatrix}. \tag{7}$$

This scattering matrix with *six independent elements* is valid if any of the following assumptions is satisfied:

1. each particle in the ensemble has a plane of symmetry (e.g. homogeneous spheroids, which include homogeneous spheres), and the particles are randomly oriented; or
2. the ensemble contains particles and their mirror particles in equal number and in random orientation; or
3. the particles are much smaller than the wavelength of light (Rayleigh limit).

The $a_1 \equiv a_1(\Theta)$ component of the Stokes scattering matrix satisfies:

$$\frac{1}{2}\int_{-1}^{1} a_1(\Theta) d(\cos\Theta) = 1 \qquad \leftarrow \qquad \textbf{normalization}. \tag{8}$$

In the scalar case this component is called:

- the *scattering phase function*:

$$p(\Theta) = p(\cos\Theta) \equiv a_1(\Theta)$$

  and it is the only one that matters if polarization effects are ignored.

The scattering phase matrix $\mathbf{M}_\mathrm{S}(\Theta)$ derived from $\mathbf{F}_\mathrm{S}(\Theta)$ pertains to the Stokes vector representation $\mathbf{I}_\mathrm{S} = [I, Q, U, V]^T$, which is related to $\mathbf{I} = [I_\ell, I_r, U, V]^T$ through $\mathbf{I}_\mathrm{S} = \mathbf{D}\mathbf{I}$, where $\mathbf{D}$ is given by Eq. (3). The corresponding scattering phase matrix is related to $\mathbf{M}_\mathrm{S}$ by:

$$\mathbf{M} = \mathbf{D}^{-1}\mathbf{M}_\mathrm{S}\mathbf{D}$$

as explained in some detail below.

Expansion in Generalized Spherical Functions (1)

The elements of $\mathbf{F}_{\mathrm{S}}(\Theta)$ in Eq. 7 can be expanded in generalized spherical functions (GSFs). To obtain the Fourier components of the scattering phase matrix from the expansion coefficients of the Stokes scattering matrix, we may use an efficient method developed by Siewert (1981, 1982), in which

- the expansion coefficients of the Stokes scattering matrix in the basis of GSFs are directly transformed into the Fourier components of the scattering phase matrix.

An advantage of this method is that

- the expressions for the Fourier components in terms of the GSFs are purely analytical.

We start by expanding the scattering phase matrix in a Fourier series as follows:

$$\mathbf{M}(\tau, u', \phi'; u, \phi) = \sum_{m=0}^{2M-1} \Big\{ \mathbf{M}^{\mathrm{cm}}(\tau, u', u) \cos m(\phi' - \phi) + \mathbf{M}^{\mathrm{sm}}(\tau, u', u) \sin m(\phi' - \phi) \Big\}. \tag{9}$$

Expansion in Generalized Spherical Functions (2)

Now use **addition theorem** for GSFs to express the Fourier expansion coefficients directly in terms of the expansion coefficients of the Stokes scattering matrix:

$$\mathbf{M}^{\mathrm{cm}}(\tau, u', u) = \mathbf{A}^m(\tau, u', u) + \tilde{\mathbf{D}}\mathbf{A}^m(\tau, u', u)\tilde{\mathbf{D}} \tag{10}$$

$$\mathbf{M}^{\mathrm{sm}}(\tau, u', u) = \mathbf{A}^m(\tau, u', u)\tilde{\mathbf{D}} - \tilde{\mathbf{D}}\mathbf{A}^m(\tau, u', u) \tag{11}$$

where $\tilde{\mathbf{D}} = \mathrm{diag}\{1, 1, -1, -1\}$. The matrix $\mathbf{A}^m(\tau, u', u)$ is given by:

$$\mathbf{A}^m(\tau, u', u) = \sum_{\ell=m}^{2N-1} \mathbf{P}^{\ell}_{m}(u)\mathbf{\Lambda}^{\ell}(\tau)\mathbf{P}^{\ell}_{m}(u'). \tag{12}$$

The elements of $\mathbf{\Lambda}^{\ell}(\tau)$ can be expressed in terms of so-called "Greek" constants:

$$\mathbf{\Lambda}^{\ell}(\tau) = \begin{pmatrix} \alpha_1^{\ell} & \beta_1^{\ell} & 0 & 0 \\ \beta_1^{\ell} & \alpha_2^{\ell} & 0 & 0 \\ 0 & 0 & \alpha_3^{\ell} & \beta_2^{\ell} \\ 0 & 0 & -\beta_2^{\ell} & \alpha_4^{\ell} \end{pmatrix}. \tag{13}$$

Expansion in Generalized Spherical Functions (3)

The elements of $\mathbf{\Lambda}^\ell(\tau)$ are the expansion coefficients of the elements of the Stokes scattering matrix in generalized functions. Thus (supressing the τ-dependence):

$$a_1(\Theta) = \sum_{\ell=0}^{2N-1} \alpha_1^\ell P_{0,0}^\ell(\cos\Theta) \tag{14}$$

$$a_2(\Theta) + a_3(\Theta) = \sum_{\ell=2}^{2N-1} (\alpha_2^\ell + \alpha_3^\ell) P_{2,2}^\ell(\cos\Theta) \tag{15}$$

$$a_2(\Theta) - a_3(\Theta) = \sum_{\ell=2}^{2N-1} (\alpha_2^\ell - \alpha_3^\ell) P_{2,-2}^\ell(\cos\Theta) \tag{16}$$

$$a_4(\Theta) = \sum_{\ell=0}^{2N-1} \alpha_4^\ell P_{0,0}^\ell(\cos\Theta) \tag{17}$$

$$b_1(\Theta) = \sum_{\ell=2}^{2N-1} \beta_1^\ell P_{0,2}^\ell(\cos\Theta) \tag{18}$$

$$b_2(\Theta) = \sum_{\ell=2}^{2N-1} \beta_2^\ell P_{0,2}^\ell(\cos\Theta). \tag{19}$$

The matrix $\mathbf{P}_m^\ell(u)$ in Eq. (12): $\mathbf{A}^m(\tau, u', u) = \sum_{\ell=m}^{2N-1} \mathbf{P}_m^\ell(u)\mathbf{\Lambda}^l(\tau)\mathbf{P}_m^\ell(u')$ is defined as:

Expansion in Generalized Spherical Functions (4)

$$\mathbf{P}_m^\ell(u) = \begin{pmatrix} P_{m,0}^\ell(u) & 0 & 0 & 0 \\ 0 & P_{m,+}^\ell(u) & P_{m,-}^\ell(u) & 0 \\ 0 & P_{m,-}^\ell(u) & P_{m,+}^\ell(u) & 0 \\ 0 & 0 & 0 & P_{m,0}^\ell(u) \end{pmatrix} \quad (20)$$

where

$$P_{m,\pm}^\ell(u) = \frac{1}{2}[P_{m,-2}^\ell(u) \pm P_{m,2}^\ell(u)]. \quad (21)$$

The functions $P_{m,0}^\ell(u)$ and $P_{m,\pm 2}^\ell(u)$ are the generalized spherical functions. Again, in the scalar case we need only the $a_1(\Theta)$ component of the Stokes scattering matrix $\mathbf{F}_\mathrm{S}(\Theta)$, and in accordance with Eq. 8, we have:

$$a_1(\Theta) = \sum_{\ell=0}^{2N-1} \alpha_1^\ell(\tau) P_{0,0}^\ell(\cos\Theta) \equiv p(\tau, \cos\Theta) = \sum_{\ell=0}^{M-1} (2\ell+1) g_\ell(\tau) P_\ell(\cos\Theta). \quad (22)$$

Thus, we have $P_{0,0}^\ell(\cos\Theta) \equiv P_\ell(\cos\Theta)$, where $P_\ell(\cos\Theta)$ is the Legendre polynomial of degree ℓ, and $\alpha_1^\ell(\tau) \equiv (2\ell+1)g_\ell(\tau)$.

Expansion in Generalized Spherical Functions (5)

For completeness we should note that:

- the expansion coefficients given above are for the scattering phase matrix $\mathbf{M}_{\rm S}$ that relates the incident and scattered Stokes vectors in the representation $\mathbf{I}_{\rm S} = [I, Q, U, V]^T$, while it is sometimes convenient
- to use the representation $\mathbf{I} = [I_\ell, I_r, U, V]^T$.

The connection between these two representations is simply

$$\mathbf{I}_{\rm S} = \mathbf{DI}$$

where $\mathbf{D}$ is given by Eq. (3), which implies that:

- the matrix $\mathbf{M}$ in the Stokes vector representation $\mathbf{I} = [I_\ell, I_r, U, V]^T$ is related to the matrix $\mathbf{M}_{\rm S}$ in the Stokes vector representation $\mathbf{I}_{\rm S} = [I, Q, U, V]^T$ as follows:

$$\mathbf{M} = \mathbf{D}^{-1}\mathbf{M}_{\rm S}\mathbf{D}.$$

Note that the rotations of the reference plane (see Eq. 4) are implicitly accounted for in the expansion method.

Examples of Scattering Phase Functions

The action of scattering particles (including molecules) on the radiance and the state of polarization of an incident radiation field can be represented as a linear operator, called

- The **scattering phase matrix**.
- The scattering phase matrix is a 4×4 matrix that connects the Stokes vector of the incident radiation to the scattered radiation.
- The elements of the scattering phase matrix depend upon the optical properties of the particles.
- The radiance of light, i.e. the first component I of the Stokes vector $\mathbf{I}_S = [I, Q, U, V]^T$, conveys information about the energy carried by the light field.
- For this purpose we need only the a_1 element of the Stokes scattering matrix, which is usually referred to as the **scattering phase function**.

Examples of Scattering Phase Functions

In many applications like:

- Heating/cooling of the medium, photodissociation of molecules, and biological dose rates

it is often permissible to ignore polarization effects. The reason is that:

- The error incurred by doing so is very small compared to errors caused by uncertainties in the input parameters to the computation, which determine the inherent optical properties of the medium.

We may therefore limit our attention to the scattering phase function if our interest lies primarily in energy transfer, although:

- **In certain remote sensing applications, the state of polarization carries additional information that may be absolutely necessary.**

Rayleigh Scattering Phase Function (1)

The elastic scattering of light by small particles or molecules, called **Rayleigh scattering**, closely follows that of an induced dipolar oscillator:

- The incident wave induces a motion of the bound electrons, which is in phase with the wave.
- The much more massive positively charged nucleus provides a 'restoring force' for the electronic motion.
- For a particle or molecule much smaller than the wavelength of light, all parts of the particle are subjected to the same value of the electric field.
- The oscillating charge radiates secondary waves. Thus: **the particle extracts energy from the wave and re-radiates it in all directions.**

If we assume that the incident radiation is unpolarized, then the normalized scattering phase function is given by (see §3.4.1)

$$p_{\text{Ray}}(\cos\Theta) = \frac{3}{3+f}(1 + f\cos^2\Theta), \tag{23}$$

where the parameter $f = \frac{1-\rho}{1+\rho}$, and ρ is the *depolarization factor* attributed to the anisotropy of the scatterer.

Rayleigh Scattering Phase Function (2)

- Expanding $p_{\rm Ray}(\cos\Theta)$ in terms of the incident and scattered polar and azimuthal angles, we find [$\cos\Theta = \underbrace{\cos\theta}_{u}\underbrace{\cos\theta'}_{u'} + \underbrace{\sin\theta}_{(1-u^2)^{1/2}}\underbrace{\sin\theta'}_{(1-u'^2)^{1/2}}\cos(\phi'-\phi)$]:

$$
\begin{aligned}
p_{\rm Ray}(u',\phi';u,\phi) &= \frac{3}{3+f}\Big[1 + fu'^2u^2 + f(1-u'^2)(1-u^2)\cos^2(\phi'-\phi) \\
&\quad + 2fu'u(1-u'^2)^{1/2}(1-u^2)^{1/2}\cos(\phi'-\phi)\Big]. \qquad (24)
\end{aligned}
$$

- The azimuthally averaged scattering phase function is found to be:

$$
\begin{aligned}
p_{\rm Ray}(u',u) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi' p_{\rm Ray}(u',\phi';u,\phi) \\
&= \frac{3}{3+f}\left[1 + fu'^2u^2 + \frac{f}{2}(1-u'^2)(1-u^2)\right] \qquad (25)
\end{aligned}
$$

where we have used: $\int_0^{2\pi} dx\cos^2 x = 1/2$; $\int_0^{2\pi} dx\cos x = 0$.

Moments of Phase Functions (1)

- We have (see Eq. 6.21):

$$p(\tau, \cos\Theta) \approx \sum_{\ell=0}^{2N-1} (2\ell+1)\chi_\ell(\tau) P_\ell(\cos\Theta) \tag{26}$$

where P_ℓ is the ℓ^{th} *Legendre polynomial*, and $\chi_\ell(\tau)$ is given by:

$$\chi_\ell(\tau) = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta) P_\ell(\cos\Theta) p(\tau, \cos\Theta). \tag{27}$$

- Further, we have (see Eq. 6.24):

$$\frac{1}{2}\int_{-1}^{1} du P_\ell(u) P_k(u) = \frac{1}{2\ell+1}\delta_{\ell k} \quad \leftarrow \quad \text{orthogonality} \tag{28}$$

$$P_\ell(\cos\Theta) = P_\ell(u') P_\ell(u) + 2\sum_{m=1}^{\ell} \Lambda_\ell^m(u') \Lambda_\ell^m(u) \cos m(\phi' - \phi) \tag{29}$$

$$\Lambda_\ell^m(u) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(u)$$

where Eq. 29 = Addition theorem, and $P_\ell^m(u)$ = associated Legendre polynomial.

Moments of Phase Functions (2)

Application of azimuthal averaging, *i.e.* $\frac{1}{2\pi}\int_0^{2\pi} d\phi \cdots$, to both sides of (26) gives

$$p(\tau, u', u) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, p(\tau, \cos\Theta) \approx \sum_{l=0}^{2N-1} (2\ell+1)\chi_\ell(\tau) P_\ell(u) P_\ell(u') \tag{30}$$

where we have made use of Eq. 29.

- From Eq. 30 it follows that:

$$\frac{1}{2}\int_{-1}^{1} p(\tau, u', u) P_k(u') du' \approx \sum_{\ell=0}^{2N-1} (2\ell+1)\chi_\ell(\tau) P_\ell(u) \frac{1}{2}\int_{-1}^{1} P_\ell(u') P_k(u') du' \tag{31}$$

which by the use of Eq. 28 leads to

$$\chi_\ell(\tau) = \frac{1}{P_\ell(u)}\,\frac{1}{2}\int_{-1}^{1} p(\tau, u', u) P_\ell(u') du'. \tag{32}$$

Thus:

- **To calculate the moments we can use the azimuthally-averaged phase function.**

Moments of Rayleigh Phase Function

- Expressing Eq. 25 in terms of **Legendre polynomials**, one may show that:

$$p_{\rm Ray}(u',u) = 1 + \frac{2f}{3+f} P_2(u)P_2(u') = P_0(u)P_0(u') + \frac{2f}{3+f} P_2(u)P_2(u'). \qquad (33)$$

- Using Eqs. 32 and 33, we obtain:

$$\chi_\ell = \frac{P_0(u)}{P_\ell(u)}\delta_{0\ell} + \frac{1}{5}\frac{2f}{3+f}\frac{P_2(u)}{P_\ell(u)}\delta_{2\ell}.$$

Thus, we get:

$$\chi_0 = 1; \qquad \chi_1 = 0,$$

and

$$\chi_2 = \frac{1}{5} \times \frac{2f}{3+f} = \frac{2f}{5(3+f)} \quad \rightarrow \quad \frac{1}{10} = 0.1 \ \text{ if } \ f = 1,$$

and

$$\chi_\ell = 0 \ \text{ for } \ \ell > 2.$$

We note that the asymmetry factor for the Rayleigh scattering phase function is $g = \chi_1 = 0$ because of the orthogonality of the **Legendre polynomials**.*

*Using symmetry arguments, one can prove that $g = 0$ for any even function of $\cos\Theta$, that is, for any term in a scattering phase function that is symmetric around $\cos\Theta = 90°$.

The Mie-Debye Scattering Phase Function (1)

Scattering in planetary media is caused by molecules and particulate matter:

- If the size of the scatterer is small compared to the wavelength as is the case for scattering of solar radiation by molecules, then the scattering phase function is only mildly anisotropic.
- Such a scattering phase function poses no special problem when solving the radiative transfer equation.
- On the other hand, scattering of solar radiation by larger particles is characterized by strong forward scattering with a so-called *diffraction peak* in the forward direction.

To understand why these larger dielectric particles have a preference for scattering in the forward direction, we may:

- consider a simplified model of a scattering particle which is not small compared with the wavelength.

The Mie-Debye Scattering Phase Function (2)

- Recall first that: **a Rayleigh scatterer can be understood in terms of the emission induced from a single excited dipole.**
- Even a very small particle might still be composed of many thousands of elementary dipoles, BUT:
  - the emission from this array of dipoles all add together **coherently** if the wavelength of the incoming radiation is large compared to the size of the particle.
- WHY? Because all the oscillators are in phase, since they are subjected to essentially the same electric field. Thus: **the radiated pattern is exactly the same as that of a single dipole.**
- If the particle size is comparable to, or larger than, the wavelength, all parts of the dipole are no longer in phase.

The Mie-Debye Scattering Phase Function (3)

In this case one finds that:

- **The scattered wavelets in the forward direction are always in phase**;
- those emitted in the backward direction usually suffer some mutual cancellation. Wavelets emitted in other directions will also be partially interferring.
- This very simplified picture explains the predominance of the forward "diffraction" peak in large-particle scattering.
- The *Mie-Debye* theory has been refined and developed by hundreds of investigators. Although the mathematical foundation is complete, its numerical implementation has proven to be very challenging.
- Progress in developing fast and accurate computer algorithms continues to the present day.

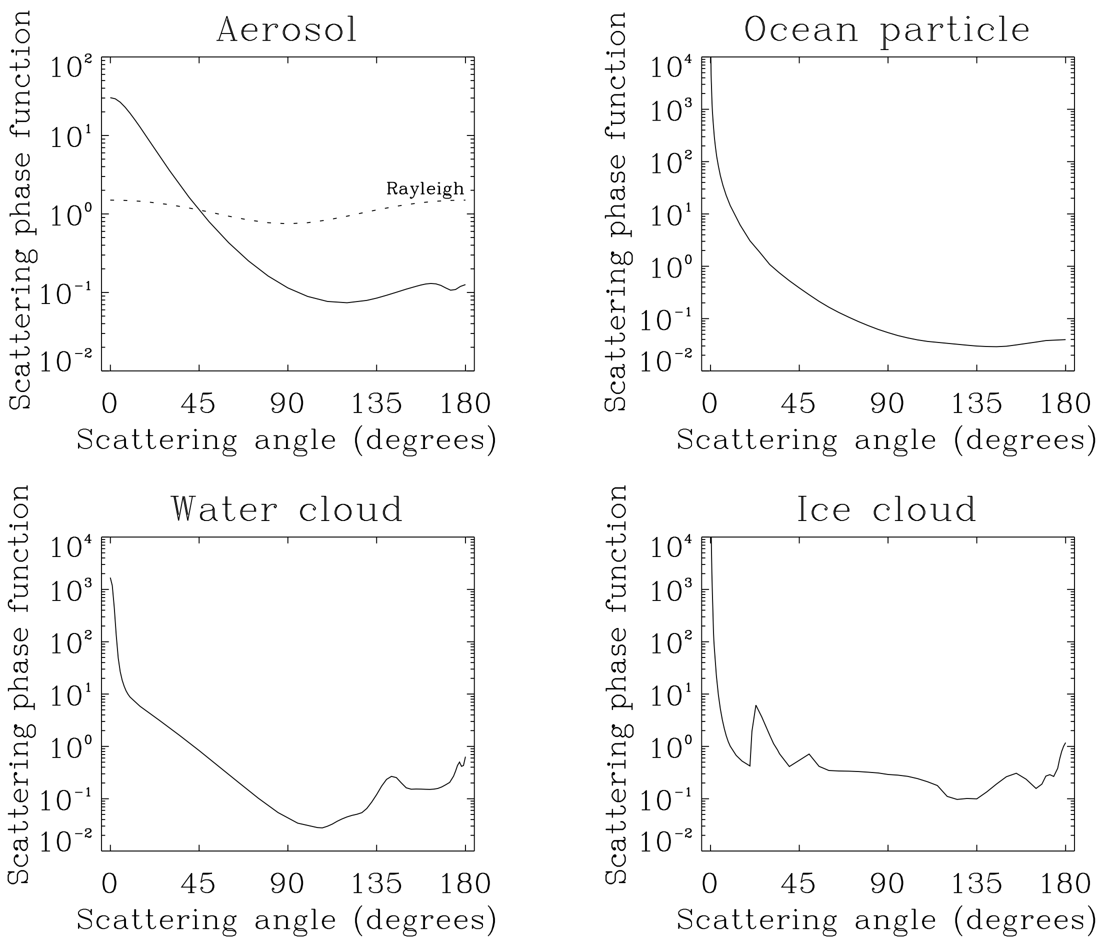


Figure 2: **Illustration of phase functions occurring in planetary media. Shown are phase functions for: molecular (Rayleigh) scattering, and aerosol particles (upper left); hydrosols (upper right); cloud droplets (lower left); and ice crystals (lower right).**

The Henyey-Greenstein Scattering Phase Function (1)

A one-parameter phase function first proposed by the astronomers Henyey and Greenstein in 1941 is

$$p_{\mathrm{HG}}(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}}. \tag{34}$$

- This function has no physical basis, and should be considered as a one-parameter analytic fit to an actual phase function.
- It should not be used except when the fit is reasonably good. However:
- as far as the radiative transfer is concerned, the requirement of 'reasonableness' is not very strict, because the multiple scattering process tends to smooth out irregularities present in the more accurate function.
- A remarkable feature of the HG function is the fact that the *Legendre-polynomial* coefficients are simply:

$$\chi_l = (g)^\ell.$$

The Henyey-Greenstein Scattering Phase Function (2)

This feature explains its popularity: only the first moment of the phase function, *i.e.*, the asymmetry factor g must be specified. Thus, the *Legendre polynomial* expansion of the H-G phase function is given simply by:

$$p_{\mathrm{HG}}(\cos\Theta) = 1 + 3g\cos\Theta + 5g^2 P_2(\cos\Theta) + \cdots = \sum_{\ell=0}^{\infty}(2\ell+1)g^{\ell}P_{\ell}(\cos\Theta).$$

Also, the *Henyey-Greenstein* phase function yields:
complete forward scattering for $g = 1$,
isotropic scattering for $g = 0$, and
complete backward scattering for $g = -1$.
The linear combination:

$$p(\cos\Theta) = b p_{\mathrm{HG}}(g, \cos\Theta) + (1-b) p_{\mathrm{HG}}(g', \cos\Theta)$$

can be used to simulate a phase function with *both* a forward and a backward scattering component ($g > 0$ and $g' < 0$). Here, $0 < b < 1$ and g and g' are usually different.

The Fournier–Forand Scattering Phase Function (1)

Measurements have shown that the *particle size distribution* (PSD) in oceanic water can be accurately described by a power law (*Junge distribution*)

$$n(r) = C(\xi, r_1, r_2)/r^{\xi} \qquad \text{where}$$

- $n(r)$ is the number of particles per unit volume per unit bin width,
- r [μm] is the radius of the assumed spherical particles, and r_1 and r_2 denote the smallest and largest particle size, respectively.
- The normalization constant $C(\xi, r_1, r_2)$ [$\text{cm}^{-3} \cdot \mu\text{m}^{\xi-1}$] is called the *Junge coefficient*, and the PSD slope ξ typically varies between 3.0 and 5.0.

Fournier and Forand derived an analytic expression for the scattering phase function of oceanic water (the FF scattering phase function) given by

$$\begin{aligned} p_{\text{FF}}(\Theta) \;&=\; \frac{1}{4\pi(1-\delta)^2\delta^{\nu}}\left\{\nu(1-\delta) - (1-\delta^{\nu}) + \frac{4}{\tilde{u}^2}[\delta(1-\delta^{\nu}) - \nu(1-\delta)]\right\} \\ &+\; \frac{1-\delta_{180}^{\nu}}{16\pi(\delta_{180}-1)\delta_{180}^{\nu}}[3\cos^2\Theta - 1], \end{aligned} \tag{35}$$

where $\nu = 0.5(3-\xi)$ and $\delta_{180} = \delta(\Theta = 180°) = \frac{4}{3(m_{\text{r}}-1)^2}$.

The Fournier–Forand Scattering Phase Function (2)

The parameter

$$\delta \equiv \delta(\Theta) = \frac{\tilde{u}^2(\Theta)}{3(m_{\rm r}-1)^2}, \qquad \tilde{u}(\Theta) = 2\sin(\Theta/2).$$

- Note that in addition to the scattering angle, Θ, the FF *scattering phase function* depends also on the real part of the refractive index of the particle relative to water, $m_{\rm r}$, and the *slope parameter*, ξ, characterizing the PSD.

Setting $x = -\cos\Theta$, and integrating the FF scattering phase function over the backward hemisphere, one obtains the *backscattering ratio* (Eq. 6.23)

$$\begin{aligned} b_{\rm FF} &= \frac{1}{2}\int_{\pi/2}^{\pi} p_{\rm FF}(\cos\Theta)\sin\Theta d\Theta = \frac{1}{2}\int_0^1 p_{\rm FF}(-x)\,dx \\ &= 1 - \frac{1-\delta_{90}^{\nu+1} - 0.5(1-\delta_{90}^{\nu})}{(1-\delta_{90})\delta_{90}^{\nu}}, \end{aligned} \tag{36}$$

where $\delta_{90} = \delta(\Theta = 90°) = \frac{4}{3(m_{\rm r}-1)^2}\sin^2(45°) = \frac{2}{3(m_{\rm r}-1)^2}$. Equation 36 can be solved for ν in terms of $b_{\rm FF}$ and δ_{90}, implying that ν and thus ξ can be determined if $m_{\rm r}$ and $b_{\rm FF}$ are specified. Hence, $p_{\rm FF}(\Theta)$ can be evaluated from a measured value of $b_{\rm FF}$ if $m_{\rm r}$ is known.

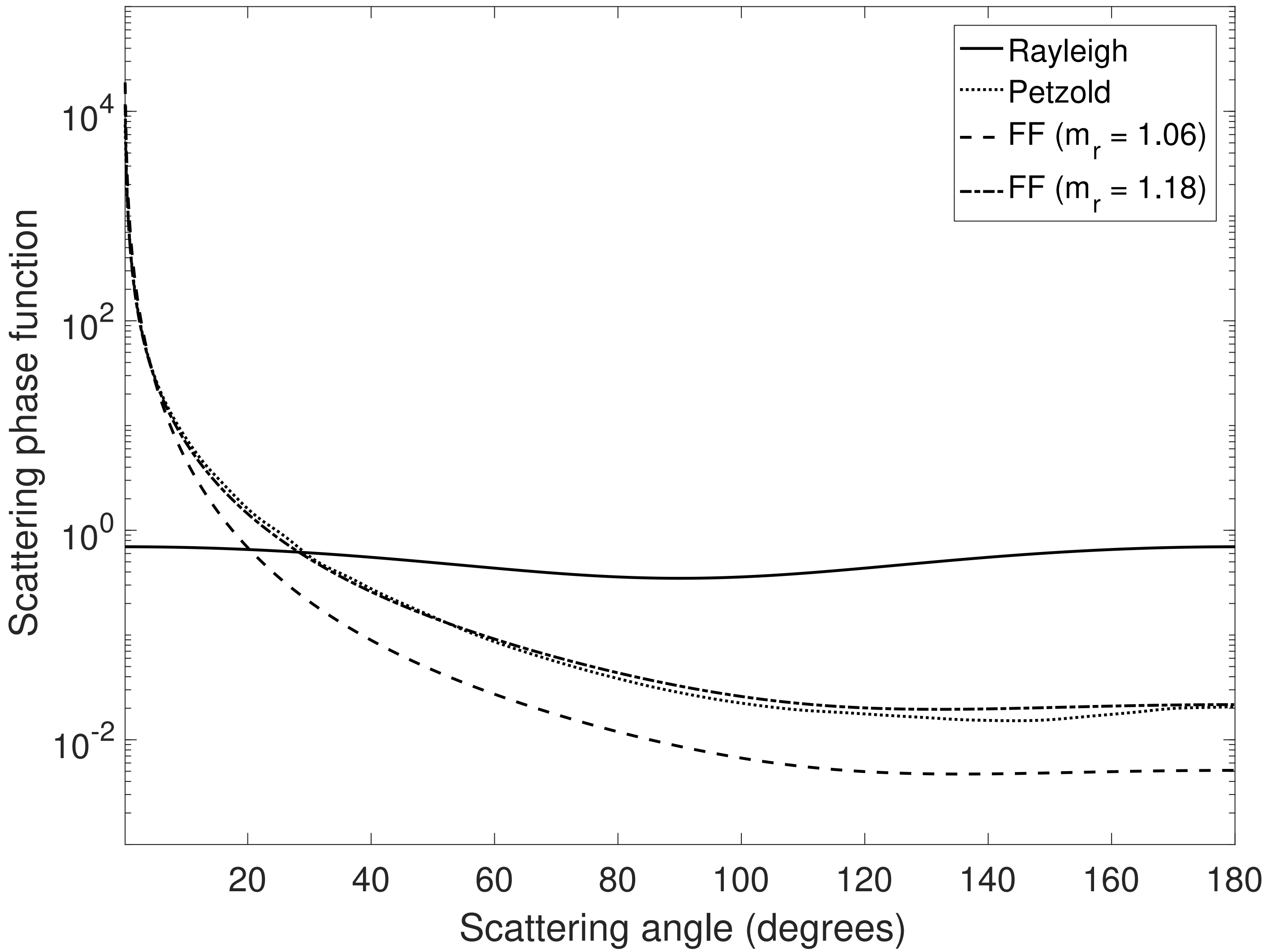


Figure 3: The Rayleigh (Eq. 23), the FF (Eq. 35), and the Petzold scattering phase functions.

The Petzold Scattering Phase Function

Scattering phase functions measured by Petzold (1972) have been widely used by ocean optics researchers. They are discussed by Mobley (1994), who tabulated scattering phase functions for clear ocean, coastal ocean, and turbid harbor waters.

- An "average" *Petzold scattering phase function*, which has an asymmetry factor $g = 0.9223$ and a backscattering ratio $b_{\rm FF} = 0.019$, is shown in Fig. 3 together with the *Rayleigh scattering phase function* and the FF *scattering phase function*.
- For the FF scattering phase function, the power law slope was set to $\xi = 3.38$, but results for **two different values** of the real part of the refractive index are shown: $m_{\rm r} = 1.06$ and $m_{\rm r} = 1.18$. These values yield an asymmetry factor $g = 0.9693$ and a backscattering ratio $b_{\rm FF} = 0.0067$ for $m_{\rm r} = 1.06$ and $g = 0.9160$ and $b_{\rm FF} = 0.022$ for $m_{\rm r} = 1.18$.
- Noting the similarity between the FF scattering phase function for $m_{\rm r} = 1.18$ and the average Petzold scattering phase function, we conclude that
- **the average Petzold scattering phase function is more suitable for mineral-dominated waters than for pigment-dominated waters.**

Scaling Transformations Useful for Anisotropic Scattering (1)

The solution of the radiative transfer equation for strongly forward-peaked scattering is notoriously difficult:

- An accurate expansion of the scattering phase function may require several hundred terms for a typical cloud or hydrosol scattering phase function.

As we shall see, most methods of solving the RTE start by approximating the integral term by a finite sum:

- The number of terms in this sum is usually of the same order ($2N$) as the number of terms necessary to get a good *Legendre polynomial* representation of the scattering phase function.
- This approach may lead to such a large system of equations that the solution becomes impractical even on modern computers.

Scaling Transformations Useful for Anisotropic Scattering (2)

To circumvent this difficulty with strongly forward-peaked scattering:

- So-called **scaling transformations** have been invented.
- The motivation is to transform a radiative transfer equation with a strongly forward-peaked scattering phase function into a more tractable problem with a scattering phase function that is much less anisotropic.

The pronounced forward scattering by cloud droplets becomes even more extreme if we plot the scattering phase function as a function of the **cosine** of the scattering angle (instead of the scattering angle):

- The forward scattering peak takes on the resemblance of a Dirac δ-function when plotted versus *cosine* of the scattering angle.
- This behavior suggests that it would be useful to treat photons scattered within the sharp forward peak as *un*scattered, and truncate this peak from the scattering phase function.

Scaling Transformations Useful for Anisotropic Scattering (3)

- We start by assuming that the forward scattering peak can be represented by a Dirac δ-function, while the remainder of the scattering phase function is expanded in **Legendre polynomials** as usual. Thus, we set:

$$\begin{aligned}\hat{p}_{\delta-\mathrm{M}}(\cos\Theta) &\equiv 2f\delta(1-\cos\Theta)+(1-f)\sum_{\ell=0}^{M-1}(2\ell+1)\hat{\chi}_\ell P_\ell(\cos\Theta)\\ &= \hat{p}_{\delta-\mathrm{M}}(u',\phi';u,\phi)=4\pi f\delta(u'-u)\delta(\phi'-\phi)\\ &+ (1-f)\sum_{\ell=0}^{2M-1}(2\ell+1)\hat{\chi}_\ell\left\{\sum_{m=0}^{\ell}\Lambda_\ell^m(u')\Lambda_\ell^m(u)\cos m(\phi'-\phi)\right\}\end{aligned} \tag{37}$$

- $\delta(1-\cos\Theta)=2\pi\delta(u'-u)\delta(\phi'-\phi)$, and
- f $(0\le f\le 1)$ is a dimensionless parameter to be determined by a fit to an actual scattering phase function.

Scaling Transformations Useful for Anisotropic Scattering (4)

We shall refer to this transformation as the $\delta-$M method.

Note that:

- If $f = 0$ we retain the usual Legendre polynomial expansion and $\hat{\chi}_\ell \equiv \chi_\ell$.

For simplicity we consider the azimuthally-averaged radiative transfer equation below.

- We first find a general expression for the azimuthally-averaged scaled scattering phase function $[2\delta(1 - \cos\Theta) = 4\pi\delta(u' - u)\delta(\phi' - \phi)]$:

$$\begin{aligned} \hat{p}_{\delta-\mathrm{M}}(u', u) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \hat{p}(\cos\Theta) \\ &= 2f\delta(u' - u) + (1 - f)\sum_{\ell=0}^{M-1}(2\ell + 1)\hat{\chi}_\ell P_\ell(u')P_\ell(u). \end{aligned} \tag{38}$$

The δ-isotropic approximation (1)

Remove forward-scattering peak, and approximate remainder of the scattering phase function (PF) by a constant, keeping only the $\ell = 0$ term in Eq. 38 (isotropic scattering). Then, the ϕ-averaged PF becomes:

$$\hat{p}_{\delta-\mathrm{iso}}(u', u) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \hat{p}(\cos\Theta) = 2f\delta(u' - u) + (1 - f). \tag{39}$$

Use of this scattering phase function in the ϕ-averaged RTE (Eq. 6.33) yields:

$$\begin{aligned} u\frac{dI(\tau, u)}{d\tau} &= I(\tau, u) - \frac{\varpi}{2} \int_{-1}^{1} du' p(u', u) I(\tau, u') \\ &= I(\tau, u) - \varpi f I(\tau, u) - \frac{\varpi(1 - f)}{2} \int_{-1}^{1} du' I(\tau, u') \end{aligned} \tag{40}$$

$$\boxed{u\frac{dI(\hat{\tau}, u)}{d\hat{\tau}} = I(\hat{\tau}, u) - \frac{\hat{\varpi}}{2} \int_{-1}^{1} du' I(\hat{\tau}, u')} \tag{41}$$

$$d\hat{\tau} \equiv (1 - \varpi f) d\tau; \qquad \hat{\varpi} \equiv \frac{(1 - f)\varpi}{1 - \varpi f}. \tag{42}$$

The δ-isotropic approximation (2)

For simplicity we have ignored the source term $Q(\tau, u) \equiv S^*(\tau, u) + (1 - \varpi)B(\tau)$. Because we have divided by $1 - \varpi f$ this term simply becomes:

$$\hat{Q}(\hat{\tau}, u) = Q(\tau, u)/(1 - \varpi f).$$

Finally, to complete the scaling, we ask:

- How do we specify f, the strength of the forward scattering peak?
- There is no unique choice, but it should depend in some simple way on the asymmetry factor, g.

Since $\chi_1 = g$ is the first-moment of the unscaled scattering phase function: equate the *first moment* of $p_{\rm acc}$ (the accurate scattering phase function) to the first moment of the scaled scattering phase function $\hat{p}$:

$$\hat{\chi}_1 = \frac{1}{P_1(u)} \frac{1}{2} \int_{-1}^{1} du' u' \hat{p}_{\delta-{\rm iso}}(u', u) = f. \tag{43}$$

This result follows from substitution of Eq. 39 in Eq. 43 and carrying out the integration. Our matching requirement then yields: $f = \chi_1 = g$.

The δ-isotropic approximation (3)

Note that:

- The δ-isotropic approximation is sometimes referred to as the **transport approximation**.
- Use of it in the RTE leads to a RTE with isotropic scattering, but with a scaled optical depth

$$d\hat{\tau} = (1 - \varpi g) d\tau$$

and a scaled single-scattering albedo

$$\hat{\varpi} = (1 - g)\varpi / (1 - \varpi g).$$

- Thus, **the RTE with strongly anisotropic scattering is reduced to a RTE with isotropic scattering which is much easier to handle numerically.**
- These particular scaling transformations of the optical depth and the single-scattering albedo are sometimes referred to as **similarity relations**.

The δ-TTA approximation

A better approximation results from representing the remainder of the PF by two terms as follows (setting $M = 1$ in Eq. 38):

$$\hat{p}_{\delta-\mathrm{TTA}}(u', u) = 2f\delta(u' - u) + (1 - f)\sum_{\ell=0}^{1}(2\ell + 1)\hat{\chi}_\ell P_\ell(u')P_\ell(u). \tag{44}$$

Substitution of this PF into the azimuthally-averaged RTE yields:

$$u\frac{dI(\hat{\tau}, u)}{d\hat{\tau}} = I(\hat{\tau}, u) - \frac{\hat{\varpi}}{2}\sum_{\ell=0}^{1}(2\ell + 1)\hat{\chi}_\ell P_\ell(u)\int_{-1}^{1} du' P_\ell(u')I(\hat{\tau}, u') \tag{45}$$

where $d\hat{\tau}$ and $\hat{\varpi}$ are defined in Eq. 42.

Again, by matching moments of the approximate and accurate scattering phase functions we find:

$$\hat{\chi}_1 \equiv \hat{g} = \frac{\chi_1 - f}{1 - f} = \frac{g - f}{1 - f}; \qquad f = \chi_2.$$

Remarks on low-order scaling approximations (1)

We may now ask:

- What (if anything) has been gained by the scaling?
- First, we note that the TTA makes the replacement $\chi_\ell = 0$ for $\ell \geq 2$. Thus, **all the higher order moments**, which may contribute substantially to the phase function if it is strongly anisotropic, **have been set to zero.**
- When we discuss the general case (arbitrary M) below, we will find that the $\delta-$TTA approximation is equivalent to setting $\chi_\ell = \chi_2$ for $\ell \geq 2$.
- Thus, for strongly anisotropic phase functions, $\chi_3 = \chi_2$, which demonstrates the advantage of the TTA scaling transformation, i.e.,

  $\Longrightarrow$ **the third moment and higher moments are set equal to the second moment, which is generally expected to be better than setting them to zero.**

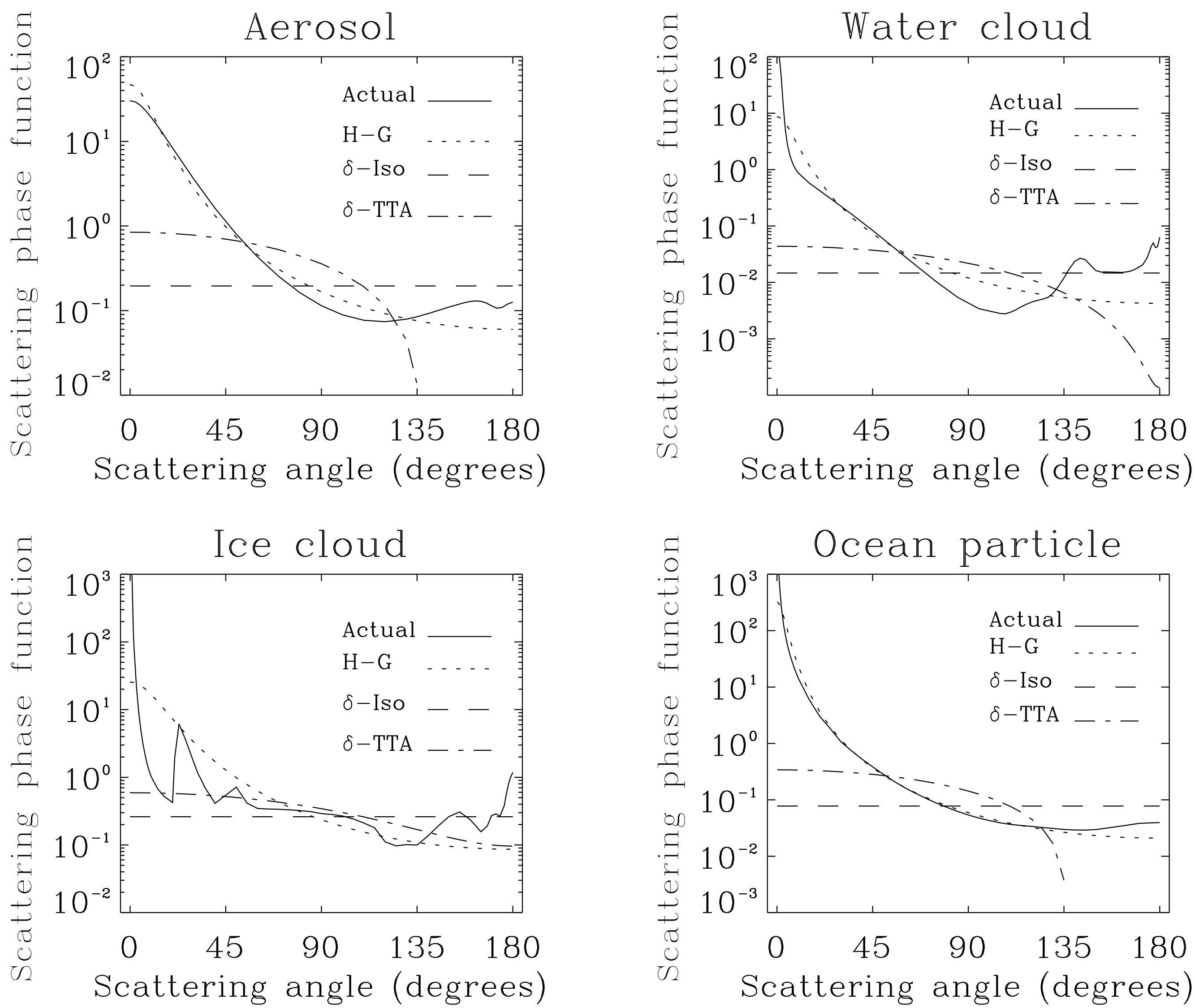


Figure 4: **Illustration of actual and δ−M scaled scattering phase functions of aerosol particles, cloud droplets, ice crystals, and hydrosols.**

Remarks on low-order scaling approximations (2)

The two-term approximation is commonly used in connection with the so-called **two-stream** and **Eddington approximations** to be discussed in Chapter 7:

- In these approximations the RTE is replaced by two coupled, first order differential equations which are easily solved analytically.
- In the two-stream case, the two coupled equations are obtained by replacing the integral (multiple-scattering term) by just two (quadrature) terms or "streams," i.e., we set $\int_{-1}^{1} du\, I(\tau, u) \approx I^{+}(\tau) + I^{-}(\tau)$.
- In the Eddington case, we expand the radiance in **Legendre polynomials** keeping only the first two terms, i.e., we set $I(\tau, u) = I_0(\tau) + uI_1(\tau)$. Insertion into the RTE and use of the first two moments lead to two coupled equations.
- Note that, as a rule of thumb, **it is customary to keep the number of terms in the expansion of the scattering phase function equal to the number of quadrature terms (or expansion terms for the radiance)**.

Remarks on low-order scaling approximations (3)

- Hence, a two-term expansion of the scattering phase function leads naturally to the two-stream (or Eddington) approximation.
- However, it is possible to use the "exact" phase function even in the two-stream approximation. The *"rule of thumb"* merely implies that **there may not be much to gain** from using a very accurate representation of the scattering phase function if the approximation for solving the RTE is much cruder.
- Finally, if we use the HG scattering phase function with the δ-TTA, then $f = g^2$, and therefore $\hat{g} = g/(1+g)$: $\Longrightarrow$ $0 \leq \hat{g} \leq 0.5$ when $0 \leq g \leq 1$.
- Thus, the $\delta-$TTA applies to a range of $\hat{g}$ $(\hat{g} < 0.5)$ for which the TTA has been shown to be reasonably accurate.
- However, we must require $\hat{g} \equiv \hat{\chi}_1 < 1/3$ to guarantee that the scaled PF [i.e. $\hat{p}_{\delta-\mathrm{TTA}}(\cos\Theta) = (1-f)(1+3\hat{g}\cos\Theta)$] is positive for all scattering angles:
- **we may obtain unphysical results (e.g. negative reflectance) unless $\hat{g} < 1/3$ or $g < 1/2$.**

The δ–M approximation: Arbitrary M (1)

We generalize the method to include an arbitrary number of terms for the remainder of the PF in Eq. 37. Substituting Eq. 38 into Eq. 40, we find

$$u\frac{dI(\hat{\tau},u)}{d\hat{\tau}} = I(\hat{\tau},u) - \frac{\hat{\varpi}}{2}\sum_{l=0}^{M-1}(2l+1)\hat{\chi}_l P_l(u)\int_{-1}^{1} du' P_l(u') I(\hat{\tau},u') \qquad (46)$$

where $d\hat{\tau}$ and $\hat{\varpi}$ are defined in Eq. 42. As before:

- We set the expansion coefficients $\hat{\chi}_\ell$ equal to the moments, χ_ℓ, of the accurate (unscaled) scattering phase function by equating moments:

$$\chi_\ell = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta) p_{\mathrm{acc}}(\cos\Theta) P_\ell(\cos\Theta)$$
$$\hat{\chi}_\ell = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta) \hat{p}_{\delta-\mathrm{M}}(\cos\Theta) P_\ell(\cos\Theta)$$

where p_{acc} denotes the accurate value for p. This procedure leads to:

$$\chi_\ell = f + (1-f)\hat{\chi}_\ell \qquad \text{or} \qquad \hat{\chi}_\ell = \frac{\chi_\ell - f}{1-f}.$$

The $\delta-$M approximation: Arbitrary M (2)

Note that:

- If we set $f = 0$, then $d\hat{\tau} = d\tau$, $\hat{\varpi} = \varpi$, and $\hat{\chi}_\ell = \chi_\ell$: **the scaled** equation reduces to the **unscaled** one as it should.
- We determine f by setting $f = \chi_M$ (truncation), which is clearly a generalization of the procedure used for $M = 1$.
- Setting $\hat{\chi}_\ell = 0$ for $\ell \geq M$ is equivalent to replacing χ_ℓ with χ_M for $\ell \geq M$: **While the ordinary Legendre polynomial expansion of order M sets $\chi_\ell = 0$ for $\ell \geq M$, the $\delta-$M method makes the replacement $\chi_\ell = \chi_M$ for $\ell \geq M$.**
- Finally we note that the error in the scattering phase function representation incurred by using the $\delta-$M method is:

$$p_{\rm acc}(\cos\Theta) - \hat{p}_{\delta-{\rm M}}(\cos\Theta) = \sum_{\ell=M+1}^{\infty} (2\ell+1)(\chi_\ell - \chi_M) P_\ell(\cos\Theta).$$

Example: The δ-Henyey-Greenstein approximation (δ–HG)

- In this case we have $\hat{p}(\cos\Theta) = 2f\delta(1-\cos\Theta) + (1-f)p_{\rm HG}(\cos\Theta)$ where $p_{\rm HG}(\cos\Theta) = \sum_{\ell=0}^{\infty}(2\ell+1)g^{\ell}P_{\ell}(\cos\Theta)$.
- By matching the first two moments of this scattering phase function ($\hat{\chi}_1$ and $\hat{\chi}_2$) to the actual scattering phase function, we find:

$$\hat{\chi}_1 = f + (1-f)g = \chi_1$$

$$\hat{\chi}_2 = f + (1-f)g^2 = \chi_2.$$

Solving for g and f we find:

$$g = \frac{\chi_1 - \chi_2}{1-\chi_1}; \qquad f = \frac{\chi_2 - {\chi_1}^2}{1 - 2\chi_1 + \chi_2}.$$

Mathematical and Physical Meaning of the Scaling (1)

An essential feature of the scaling is:

- To turn the unscaled problem into one in which the optical depth is reduced ($d\hat{\tau} < d\tau$), while the absorption is artificially increased ($\hat{\varpi} < \varpi$).[†]
- In addition, the scattering phase function appears considerably less anisotropic.
- BUT the new (**scaled**) RTE is of identical mathematical form to the old one: whatever "tools" are available for solving the **unscaled** equation can be applied to the **scaled equation**.
- The scaling simply makes the problem more tractable numerically because the scattering phase function of the new RTE is much less anisotropic due to the truncation of the forward scattering peak.
- Hence, we expect that many fewer terms are needed to obtain an adequate **Legendre polynomial** expansion of the scattering phase function: the scaled equation is easier to solve by numerical means.

[†]It should be noted that this decrease in the single-scattering albedo is due to a decrease in the scattering coefficient, which leads to an apparent increase in absorption, although the absorption coefficient is left unchanged.

Mathematical and Physical Meaning of the Scaling (2)

Thus:

- The $\delta-$M method is **not** a method of solution, but it makes the RTE easier to solve by available analytical and/or numerical techniques.

From the physical point of view $\delta-$M relies on the following premise:

- **Those beams that are scattered through the small angles contained within the forward peak are not scattered at all**. These beams are in fact "added back" to the original radiation field, which explains why the scaled optical depth $\hat{\tau}$ is smaller than the original τ.
- The effective asymmetry factor is also less than the original (**unscaled**) value, since the angular distribution of those beams scattered outside the forward peak is (by definition) less extreme.

What about Energy Conservation?

Consider the transmitted irradiance in the **scaled** problem:

$$F(\hat{\tau}^*) = F_\mathrm{d}(\hat{\tau}^*) + \mu_0 F^s e^{-\hat{\tau}^*/\mu_0}$$

where the 'd' subscript denotes the diffuse irradiance.

- Since $\hat{\tau}^* < \tau^*$, this means that the **scaled** directly-transmitted solar irradiance is **greater** than it is in the unscaled problem.

Because of the truncation of the scattering phase function:

- The "direct" irradiance actually contains some scattered beams of radiation travelling in very nearly the same direction as the incident beam. For example:
- the Sun's rays shining through a hazy or dusty atmosphere are spread out into a very bright blurry disk, somewhat greater than the Sun's disk itself.
- This bright blurry disk is called the Sun's **aureole**, and in fact has been used as a means of inferring the mean particle size in tropospheric haze.

What about Energy Conservation?

- A substantial fraction of the solar aureole would be included in the $\delta-$M direct irradiance.
- The scattered irradiance in this approximation would apply to those beams scattered largely outside the aureole.

Finally, we note that since the total downward irradiance must be the same whether we use scaling or not, i.e., $F^-_{\rm tot}(\hat{\tau}) = F^-_{\rm tot}(\tau)$, or

- $$F^-_{\rm d}(\hat{\tau}) + \mu_0 F^{\rm s} e^{-\hat{\tau}/\mu_0} = F^-_{\rm d}(\tau) + \mu_0 F^{\rm s} e^{-\tau/\mu_0}$$

we can always "recover" the unscaled downward diffuse irradiance by solving for $F^-_{\rm d}(\tau)$:

$$F^-_{\rm d}(\tau) = F^-_{\rm d}(\hat{\tau}) - \mu_0 F^{\rm s}(e^{-\tau/\mu_0} - e^{-\hat{\tau}/\mu_0}) \tag{47}$$

where all the quantities on the right are known.

- No such "correction" for the upward irradiance is necessary.

The δ-fit Method: Weighted SVD LS fitting (1)

The ordinary Legendre polynomial expansion of the scattering phase function is:

$$p(\cos\Theta) \approx \sum_{\ell=0}^{M-1} (2\ell+1)\chi_\ell P_\ell(\cos\Theta) \equiv \sum_{\ell=0}^{M-1} \xi_\ell P_\ell(\cos\Theta) \qquad \xi_\ell = (2\ell+1)\chi_\ell$$

$$\xi_\ell = \frac{2}{2\ell+1}\int_{-1}^{1} P_\ell(\cos\Theta) p_{\rm acc}(\cos\Theta) d(\cos\Theta)$$

and $p_{\rm acc}(\cos\Theta)$ is the actual scattering phase function. To remove the forward peak $\delta-$M replaces ξ_ℓ by $(\xi_\ell - f)/(1-f)$:

$$p_{\delta-\rm M}(\cos\Theta) = \sum_{\ell=0}^{M-1} \frac{(\xi_\ell - f)}{(1-f)} P_\ell(\cos\Theta), \qquad f = \chi_M.$$

The purpose of the $\delta-$fit Method is to compute new expansion coefficients c_ℓ which replace the ξ_ℓ, and which are constructed to minimize the error:

$$\varepsilon = \sum_i w_i \left(\frac{p'(\cos\Theta_i)}{p_{ac}(\cos\Theta_i)} - 1\right)^2 \qquad p'(\cos\Theta_i) = \sum_{\ell=0}^{\tilde{N}} c_\ell P_\ell(\cos\Theta_i)$$

where $\tilde{N}$ is the number of terms required to achieve a desired accuracy.

The δ-fit Method: Weighted SVD LS fitting (2)

Here Θ_i is the scattering angle, w_i is the weight associated with angle Θ_i, and $P_\ell(\cos\Theta_i)$ the corresponding Legendre polynomial. The expansion coefficients c_ℓ are determined by solving the least-squares fitting problem
$\partial\varepsilon/\partial c_k = 0,\ (k = 0, \ldots, \tilde{N})$:

$$\sum_i \frac{P_k(\cos\Theta_i)}{p_{\rm acc}(\cos\Theta_i)} w_i \left(\sum_{\ell=0}^{\tilde{N}} \frac{c_\ell P_\ell(\cos\Theta_i)}{p_{\rm acc}(\cos\Theta_i)} - 1 \right) = 0.$$

Truncate the PF by setting the weights for the forward-scattering angles (e.g., $\Theta < 3^\circ$) to zero, set truncation factor to $f = 1 - c_0$, and compute normalized scattering phase function:

$$p_{\delta-\rm fit}(\cos\Theta_i) = \frac{1}{1-f} p'(\cos\Theta_i).$$

Advantages of $\delta-$fit method:

- better estimation of PF at large scattering angles (where PF is small)
- forward-peak removed by setting $w_i \approx 0$ at small angles Θ_i.

The δ-fit Method: Weighted SVD LS fitting (3)

Implementation

1. evaluate $p_{\rm acc}(\cos\Theta)$ at 361 scattering angles Θ_i (each half degree; use interpolation as need be);
2. select forward peak removal angle Θ_c and set $w_i = 0$ for $\Theta_i < \Theta_c$;
3. select an initial number of terms $\tilde{N}$, and compute all required Legendre polynomials $P_\ell(\cos\Theta)$ for $\ell \le \tilde{N}$;
4. derive coefficients c_ℓ by solving set of linear equations $\partial\varepsilon/\partial c_\ell = 0$ (using SVD);
5. if ε larger than desired, increase $\tilde{N}$ and repeat previous step until ε is sufficiently small;
6. determine scaling factor $f = 1 - c_0$ and renormalize PF (divide all c_ℓ by c_0);
7. remove forward peak of PF by adjusting optical depth:
$$\hat{d}\tau = (1 - \varpi f)d\tau$$
and single-scattering albedo:
$$\hat{\varpi} = (1 - f)\varpi/(1 - \varpi f).$$

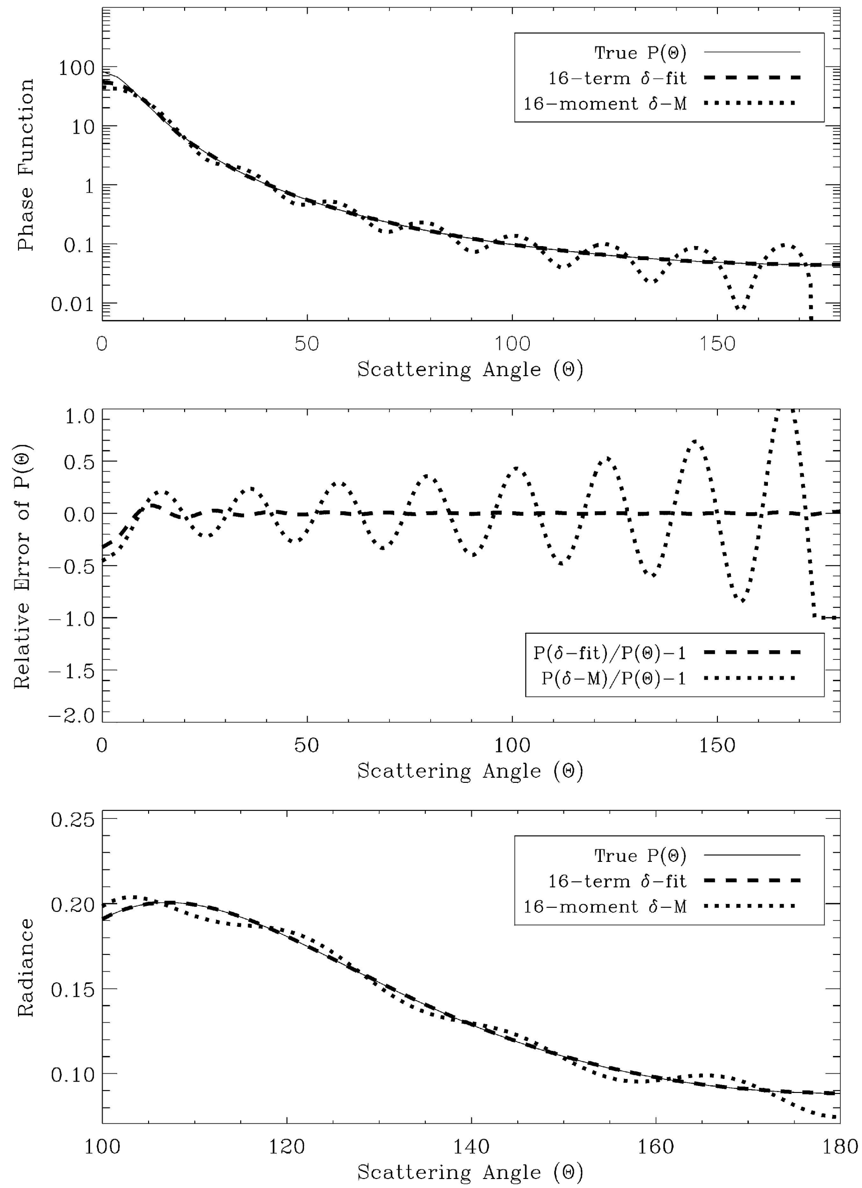


Fig. 1. H–G phase functions ($g = 0.85$) (top panel), differences among them (middle panel) and radiances (bottom panel): original phase function (solid), δ-fit: 16 term Legendre polynomial fits (dash), δ-M: 16 moments of Legendre polynomial expansion (dot).

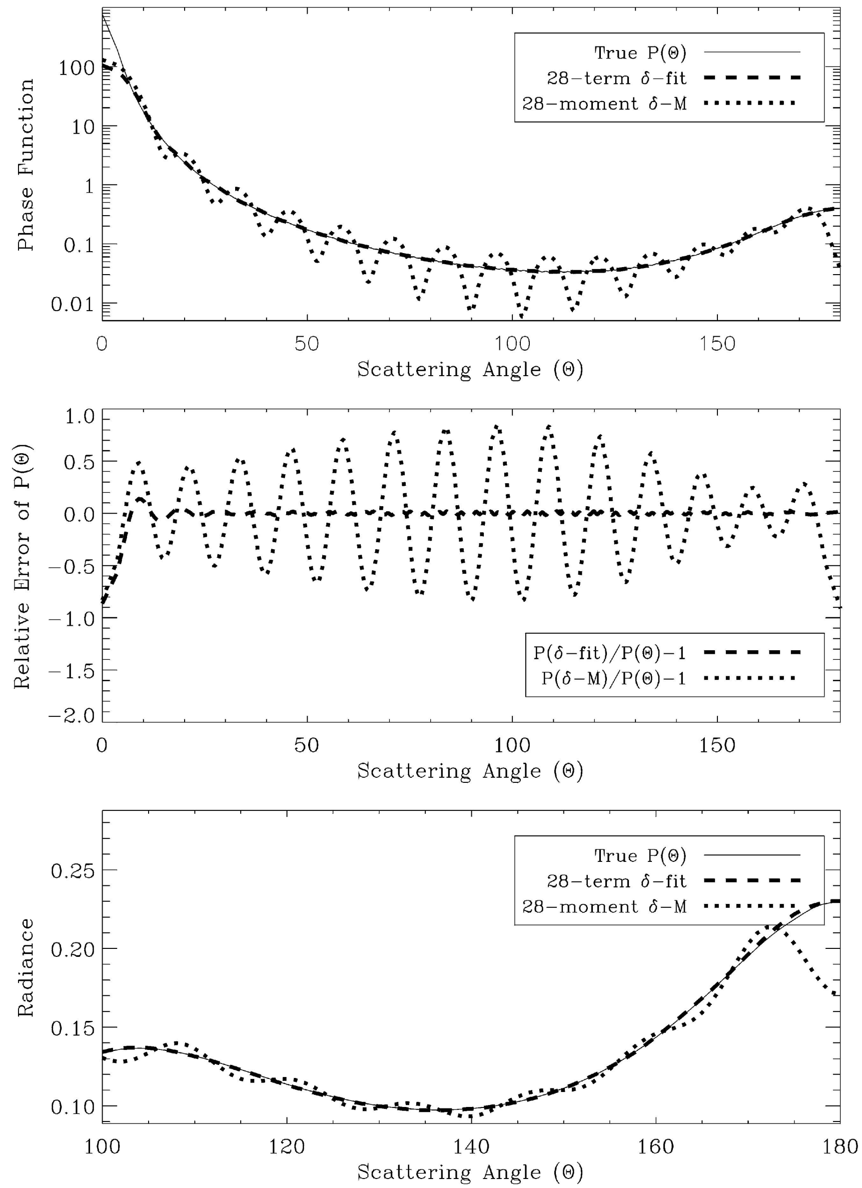


Fig. 2. Double H–G phase functions ($F = 0.98$, $g_1 = 0.92$, $g_2 = -0.7$) (top panel) differences among them (middle panel) and radiances (bottom panel): original phase function (solid), δ-fit: 28 term Legendre polynomial fits (dash), δ-M: 28 moments of Legendre polynomial expansion (dot).

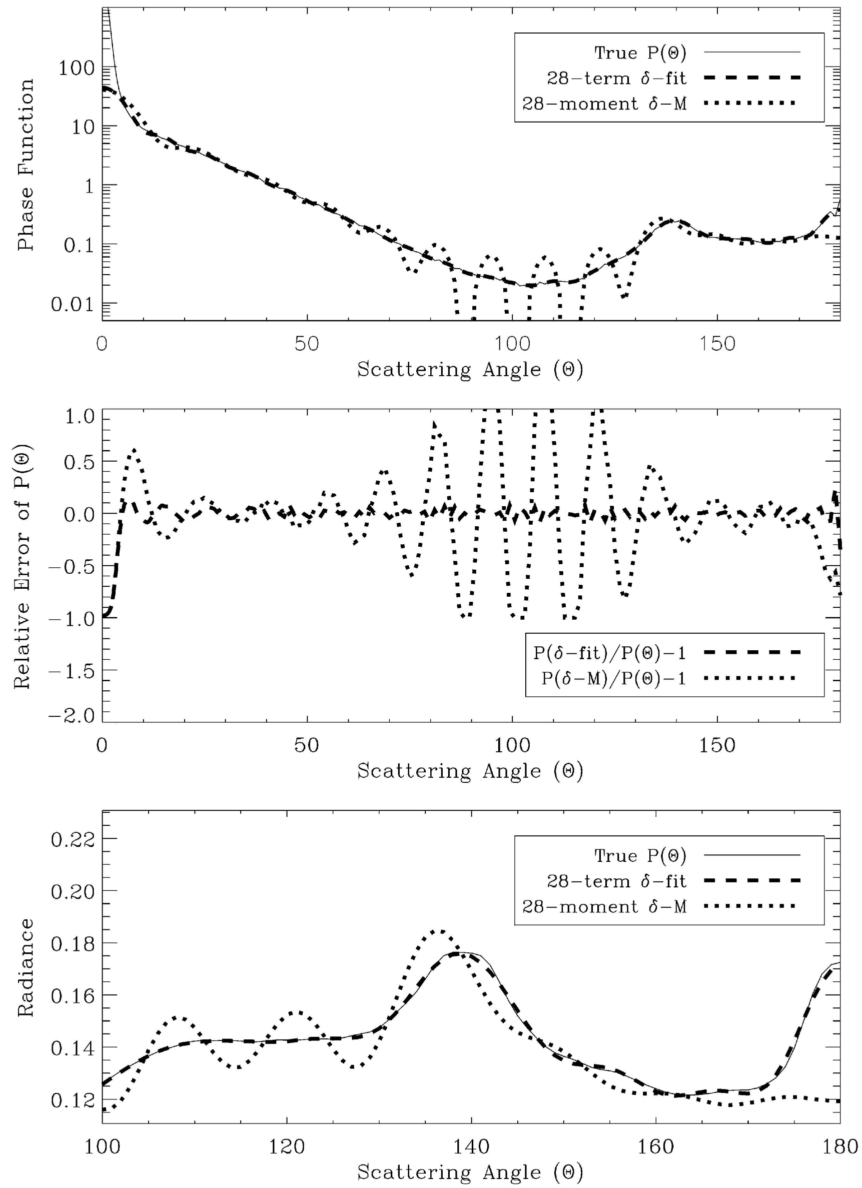
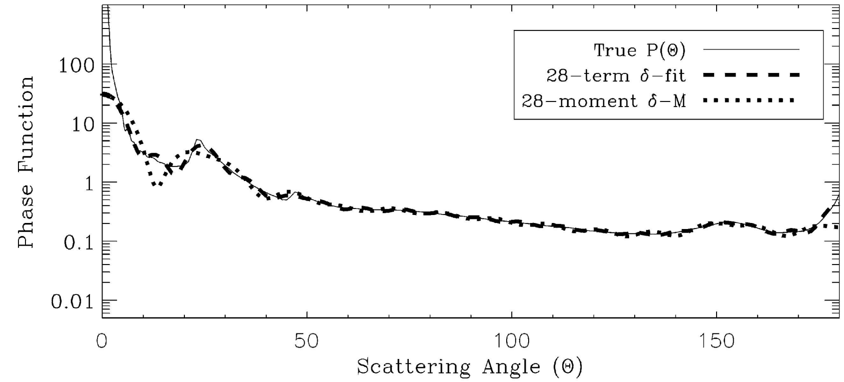
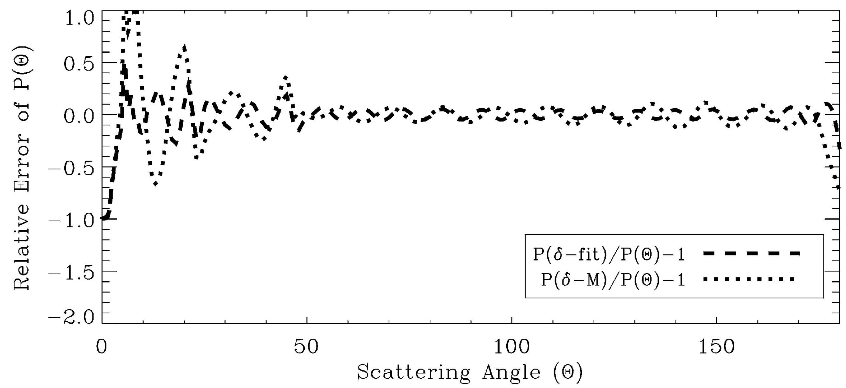


Fig. 3. Water cloud phase functions (for $\lambda = 1.6$ μm, $R_e = 10$ μm) (top panel) differences among them (middle panel) and radiances (bottom panel): original phase function (solid), δ-fit: 28 term Legendre polynomial fits (dash), δ-M: 28 moments of Legendre polynomial expansion (dot).





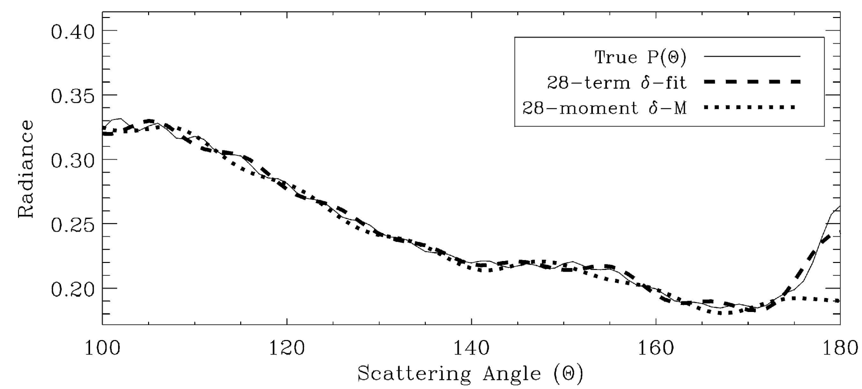


Fig. 4. Ice cloud phase functions (for $\lambda = 0.55$ μm, $D_e = 20$ μm) (top panel) differences among them (middle panel) and radiances (bottom panel): original phase function (solid), δ-fit: 28 term Legendre polynomial fits (dash), δ-M: 28 moments of Legendre polynomial expansion (dot).