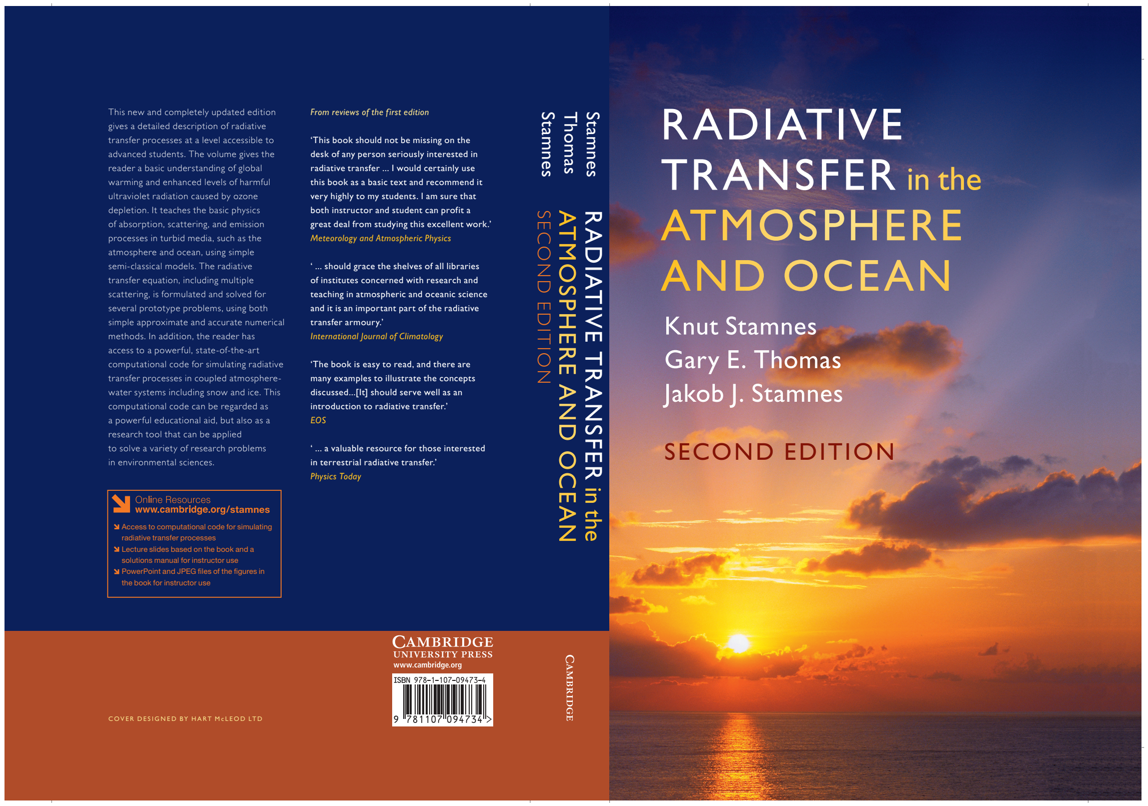
Lecture *Notes:*
Radiative Transfer Principles



Based on Chapter 5 in K. Stamnes, G. E. Thomas, and J. J. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, 2017.

Thermal Emission from a Surface (1)

- The *spectral directional emittance* is the ratio of the energy emitted by a surface of temperature $T_{\rm s}$ to the energy emitted by a blackbody at the same frequency and temperature:

$$\epsilon(\nu, \hat{\mathbf{\Omega}}, T_{\rm s}) \equiv \frac{d\omega \cos\theta I^{+}_{\nu \rm e}(\hat{\mathbf{\Omega}})}{d\omega \cos\theta B_{\nu}(T_{\rm s})} \equiv \frac{I^{+}_{\nu e}(\hat{\mathbf{\Omega}})}{B_{\nu}(T_{\rm s})}$$

- If $\epsilon = 1$ for all $\hat{\mathbf{\Omega}}$ and ν, the surface is a blackbody.
- If $\epsilon =$ constant < 1 for all $\hat{\mathbf{\Omega}}$ and ν, the surface is a gray body.
- $B_{\nu}(T_{\rm s})$ is the isotropic radiance emitted by a blackbody.

Thermal Emission from a Surface (2)

The energy emitted by the surface into the whole hemisphere is called:

- **the spectral emittance** given by:

$$
\begin{aligned}
\epsilon(\nu, 2\pi, T_\mathrm{s}) &\equiv \frac{\int_+ d\omega \cos\theta I^+_{\nu\mathrm{e}}(\hat{\mathbf{\Omega}})}{\int_+ d\omega \cos\theta B_\nu(T_\mathrm{s})} = \frac{\int_+ d\omega \cos\theta B_\nu(T_\mathrm{s})\epsilon(\nu, \hat{\mathbf{\Omega}}, T_\mathrm{s})}{\pi B_\nu(T_\mathrm{s})} \\
&= \frac{1}{\pi}\int_+ d\omega \cos\theta \epsilon(\nu, \hat{\mathbf{\Omega}}, T_\mathrm{s}). \qquad (1)
\end{aligned}
$$

Hence, the spectral emittance is the energy emitted into a hemisphere relative to a blackbody at a particular frequency.

Absorption by a Surface

- The **spectral directional absorptance** is the ratio of absorbed energy to incident energy:

$$\alpha(\nu, -\hat{\mathbf{\Omega}}', T_s) \equiv \frac{d\omega' \cos\theta' I_{\nu a}^{-}(\hat{\mathbf{\Omega}}')}{d\omega' \cos\theta' I_{\nu}^{-}(\hat{\mathbf{\Omega}}')} = \frac{I_{\nu a}^{-}(\hat{\mathbf{\Omega}}')}{I_{\nu}^{-}(\hat{\mathbf{\Omega}}')}.$$

- The energy absorbed when radiation is incident over the whole hemisphere is called the **spectral absorptance**:

$$\alpha(\nu, -2\pi, T_s) \equiv \frac{\int_{-} d\omega' \cos\theta' I_{\nu a}^{-}(\hat{\mathbf{\Omega}}')}{\int_{-} d\omega' \cos\theta' I_{\nu}^{-}(\hat{\mathbf{\Omega}}')} = \frac{\int_{-} d\omega' \cos\theta' \alpha(\nu, -\hat{\mathbf{\Omega}}', T_s) I_{\nu}^{-}(\hat{\mathbf{\Omega}}')}{F_{\nu}^{-}}.$$

- If the incident radiation is blackbody, then $I_{\nu}^{-}(\hat{\mathbf{\Omega}}') = B_{\nu}(T_s)$, $F_{\nu}^{-} = \pi B_{\nu}(T_s)$:

$$\alpha(\nu, -2\pi, T_s) = \frac{1}{\pi} \int_{-} d\omega' \cos\theta' \alpha(\nu, -\hat{\mathbf{\Omega}}', T_s).$$

Kirchoff's law for Surfaces

Consider an opaque nonblack surface within a hohlraum that is exposed to the isotropic radiance $I_\nu = B_\nu(T)$. Since the incident radiation is isotropic:

- the upward radiance must be isotropic. Since the surface is opaque:
- the upward radiance consists of an emitted component and a reflected component, which must add to yield the incident radiance: $I^+_{\nu\mathrm{e}}(\hat{\mathbf{\Omega}}) + I^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}}) = B_\nu(T_\mathrm{s})$.
- From conservation of energy the sum of the reflected and absorbed energy must be equal to the incident energy: $I^-_{\nu\mathrm{a}}(\hat{\mathbf{\Omega}}) + I^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}}) = B_\nu(T_\mathrm{s})$.
- From these two relations and the definitions of the directional emittance and absorptance, we obtain **Kirchoff's Law for an Opaque Surface**: $\alpha(\nu, -\hat{\mathbf{\Omega}}, T_\mathrm{s}) = \epsilon(\nu, \hat{\mathbf{\Omega}}, T_\mathrm{s})$.
- Note that we have assumed an isothermal enclosure in TE.
- Similarly we have in the special case of hemispherically isotropic incidence: $\alpha(\nu, -2\pi, T_\mathrm{s}) = \epsilon(\nu, 2\pi, T_\mathrm{s}) \longrightarrow$ **absorptance = emittance**.

Surface Reflection: The BRDF (1)

Consider a downward beam with radiance $I_\nu(\hat{\mathbf{\Omega}}')$ within a cone of solid angle $d\omega'$ around $\hat{\mathbf{\Omega}}'$ that is incident on a flat surface with normal $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ (see Fig. 1).

- Incident energy: $d\omega' \cos\theta' I_\nu(\hat{\mathbf{\Omega}}')$.
- Reflected radiance within $d\omega$ around $\hat{\mathbf{\Omega}}$: $dI^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}}')$.
- The **Bidirectional Reflectance Distribution Function** (BRDF) is the ratio of the reflected radiance to the incident energy:

$$\rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) \equiv \frac{dI^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}})}{d\omega' \cos\theta' I^-_\nu(\hat{\mathbf{\Omega}}')}.$$

By adding contributions to the reflected radiance in direction $\hat{\mathbf{\Omega}}$ from beams incident on the surface in all downward directions, we obtain:

$$I^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}}) = \int_- dI^+_{\nu\mathrm{r}}(\hat{\mathbf{\Omega}}) = \int_- d\omega' \cos\theta' \rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) I^-_\nu(\hat{\mathbf{\Omega}}').$$

which is **diffuse** light due to integration over the downward hemisphere.

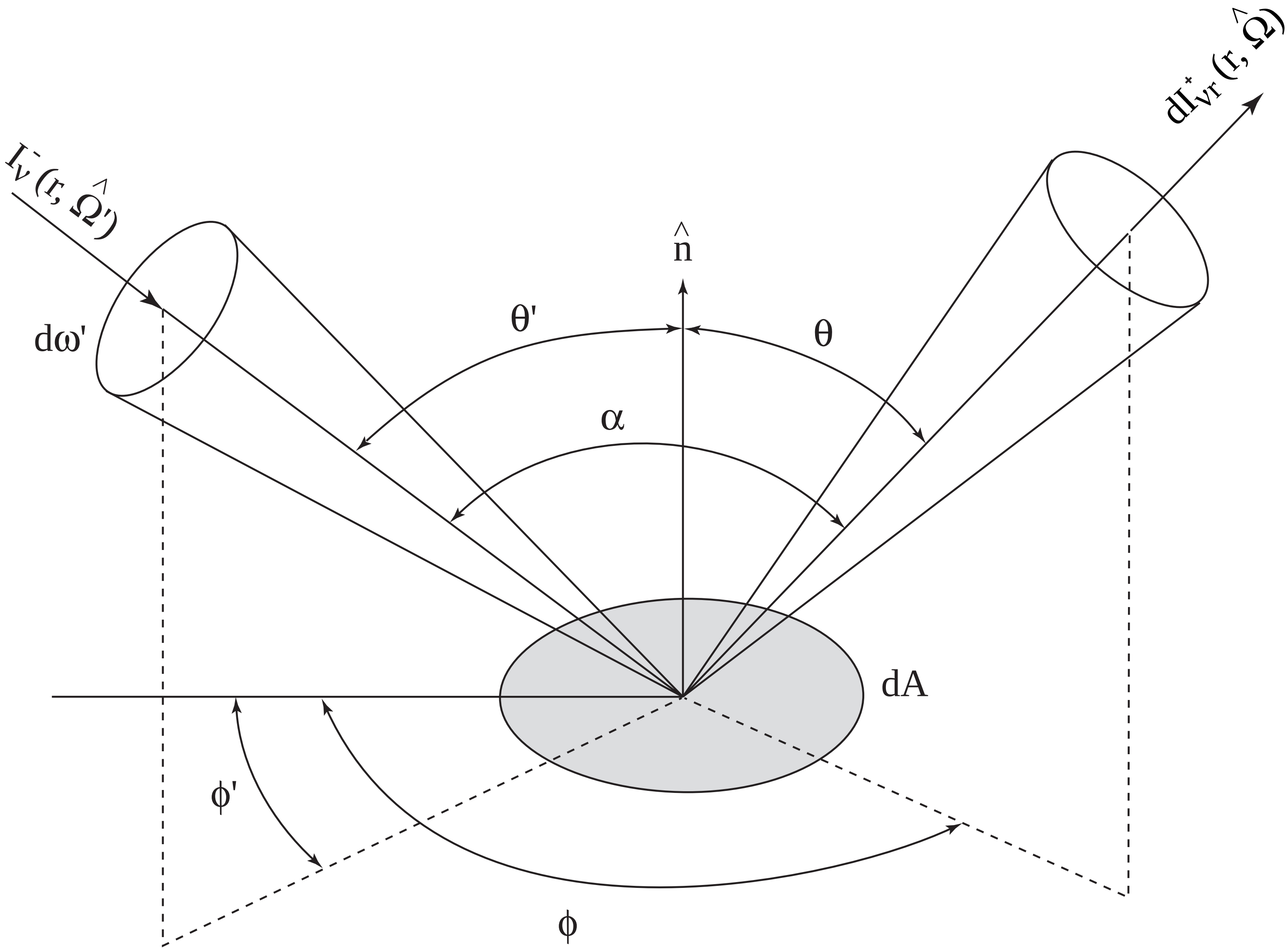


Figure 1: **Geometry and symbols for the definition of the BRDF. The angle α is the backscattering angle.**

Surface Reflection: The BRDF (2)

Note the differences between a **purely diffuse** and a **purely specular** surface. If the reflected radiance from a diffuse surface is completely uniform:

- it is called a **Lambert surface**. Examples are ground glass and matte paper.
- The BRDF of a Lambert surface is independent of both direction of incidence and direction of observation:

$$\rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) = \rho_{\mathrm{L}}(\nu)$$

  where $\rho_{\mathrm{L}}(\nu)$ = Lambert reflectance.

- For a Lambert surface the reflected radiance is:

$$I_{\nu\mathrm{r}}^{+} = \rho_{\mathrm{L}}(\nu) \int_{-} d\omega' \cos\theta' I_{\nu}^{-}(\hat{\mathbf{\Omega}}') = \rho_{\mathrm{L}}(\nu) F_{\nu}^{-}.$$

Thus, for a Lambert surface the reflected radiance is:

- independent of the observation direction and is proportional to the incident irradiance.

Surface Reflection: The BRDF (3)

For a collimated incident beam of light:

- the radiance reflected from a **Lambert surface** is proportional to the cosine of the angle of incidence (see below);
- the radiance reflected from a **specular surface** is a δ - function.

In general:

- the BRDF has a diffuse component ρ_d and a specular component ρ_s:

$$\rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) = \rho_\mathrm{s}(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) + \rho_\mathrm{d}(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}})$$

and the reflected radiance becomes:

$$I_{\nu\mathrm{r}}^{+}(\hat{\mathbf{\Omega}}) = \underbrace{\rho_\mathrm{s}(\nu, \theta) I_\nu^{-}(\theta, \phi' + \pi)}_{\text{specular component}} + \underbrace{\int_{-} d\omega' \cos\theta' \rho_\mathrm{d}(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) I_\nu^{-}(\hat{\mathbf{\Omega}}')}_{\text{diffuse component}}.$$

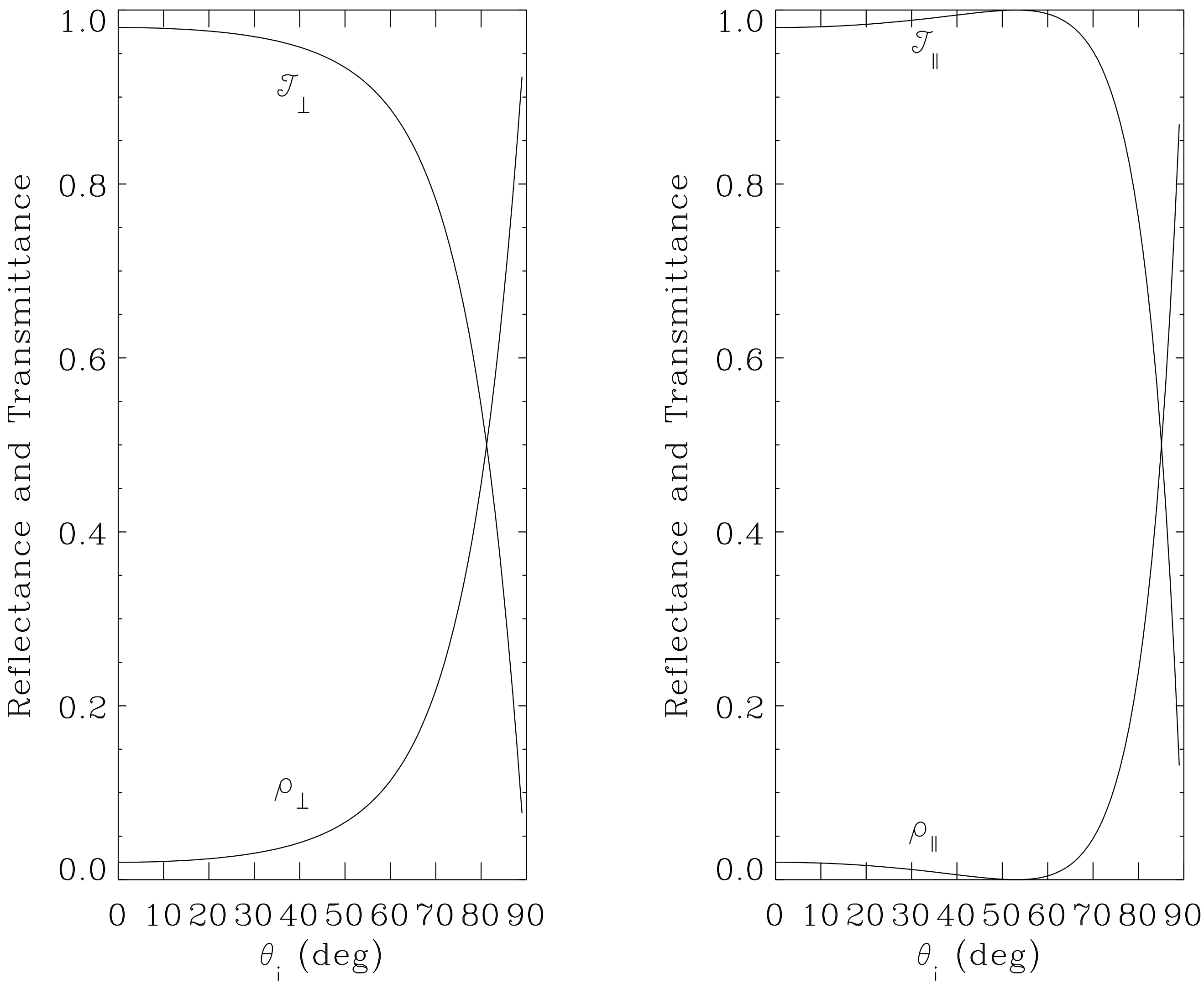


Figure 2: **The bidirectional reflectance and transmittance at a smooth air-water interface based on Fresnel's equations. The two curves show the reflectance and transmittance for the perpendicular and parallel components.**

Albedo for Collimated Incidence

- Incident radiance: $I_\nu^-(\hat{\mathbf{\Omega}}') = F_\nu^{\rm s}\delta(\cos\theta' - \cos\theta_0)\delta(\phi' - \phi_0)$.
- Diffusely reflected radiance:*

$$I_{\nu{\rm r}}^+(\hat{\mathbf{\Omega}}) = \int_- d\omega' \cos\theta' \rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) I_\nu^-(\hat{\mathbf{\Omega}}') = F_\nu^{\rm s}\cos\theta_0 \rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}}).$$

- Diffusely reflected irradiance:

$$F_{\nu{\rm r}}^+ = \int_+ d\omega \cos\theta I_{\nu{\rm r}}^+(\hat{\mathbf{\Omega}}) = F_\nu^{\rm s}\cos\theta_0 \int_+ d\omega \cos\theta \rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}}).$$

- The **reflectance or plane albedo** is the ratio of the reflected irradiance to the incident collimated beam (solar) irradiance:

$$\rho(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) \equiv \frac{F_{\nu r}^+}{F_\nu^{\rm s}\cos\theta_0} = \int_+ d\omega \cos\theta \rho(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}}).$$

- Since $\rho(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}})$ is the sum of a specular and a diffuse component:

$$\begin{aligned} \rho(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) &= \rho_{\rm s}(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) + \rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) \\ &= \int_+ d\omega \cos\theta \rho_{\rm s}(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}}) + \int_+ d\omega \cos\theta \rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}}). \end{aligned} \quad (2)$$

* $\int dx' f(x')\delta(x' - x_0) = f(x_0)$

Diffuse Incidence - Reflectance or Albedo

- If the BRDF is Lambertian, then $\rho_{\rm s} = 0$, $\rho_{\rm d}(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) = \rho_{\rm L}(\nu)$ and:

$$\rho(\nu, -\hat{\mathbf{\Omega}}_0, 2\pi) = \rho_{\rm L}(\nu) \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta = \pi \rho_{\rm L}(\nu).$$

- Consider now the diffuse sky light radiation $I_\nu^-(\hat{\mathbf{\Omega}}')$ reaching the Earth. It is distributed over the entire downward hemisphere, and the reflected irradiance is:

$$\begin{aligned} F_{\nu\rm r}^+ &= \int_+ d\omega \cos\theta I_{\nu\rm r}^+(\hat{\mathbf{\Omega}}) = \int_+ d\omega \cos\theta \int_- d\omega' \cos\theta' \rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) I_\nu^-(\hat{\mathbf{\Omega}}'). \\ &= \int_- d\omega' \cos\theta' \Big[\int_+ d\omega \cos\theta \rho(\nu, -\hat{\mathbf{\Omega}}', \hat{\mathbf{\Omega}}) \Big] I_\nu^-(\hat{\mathbf{\Omega}}') \\ &= \int_- d\omega' \cos\theta' \rho(\nu, -\hat{\mathbf{\Omega}}', 2\pi) I_\nu^-(\hat{\mathbf{\Omega}}'). \end{aligned} \quad (3)$$

- Comparing this result with our previous results for emitted and absorbed irradiances, we see that $\rho(\nu, -\hat{\mathbf{\Omega}}, 2\pi)$, $\epsilon(\nu, \hat{\mathbf{\Omega}}, T_{\rm s})$, and $\alpha(\nu, -\hat{\mathbf{\Omega}}, T_{\rm s})$ play similar roles in transforming radiance distributions into irradiances.

- For an opaque surface:

$$\alpha(\nu, -\hat{\mathbf{\Omega}}, T_{\rm s}) = 1 - \rho(\nu, -\hat{\mathbf{\Omega}}, 2\pi) \Longleftarrow \text{follows from conservation of energy.}$$

Analytical Reflectance Expressions

- **Minneart formula**: $\rho(\mu_0, \mu) = \rho_{\rm n}\mu_0^{k-1}\mu^{k-1}$
- $k = 1 \Longrightarrow$ Lambert surface
- $k = 0.5 \Longrightarrow$ "Dark" surface $\longrightarrow$ the greater the observation (nadir) angle θ the brighter the surface $\longrightarrow$ in general agreement with experiment.
- As k increases, $\rho(\mu_0, \mu)$ increases $\longrightarrow$ Disadvantage: No physical basis!
- **Principle of reciprocity:** The reflectance is unaffected by an interchange of the directions of incidence and observation:

$$\rho(\theta', \phi'; \theta, \phi) = \rho(\theta, \phi; \theta', \phi')$$

- Lommel-Seeliger formula: $\rho(\mu_0, \mu) = \frac{2\rho_{\rm n}}{\mu_0 + \mu}$:
  - $\rho_{\rm n} \equiv \rho(1, 1) =$ normal reflectance (in astronomy)
  - ρ_{min} occurs at normal viewing $(\mu = 1)$
  - ρ_{max} occurs at grazing view angle $(\mu = 0)$: consistent with Minneart formula
- As θ_0 increases from 0^0 to $90^0, \rho$ exhibits a larger increase with θ in agreement with experiments.
- Lommel-Seeliger formula applies well to dark surfaces ($\rho_{\rm n} < 0.3$).

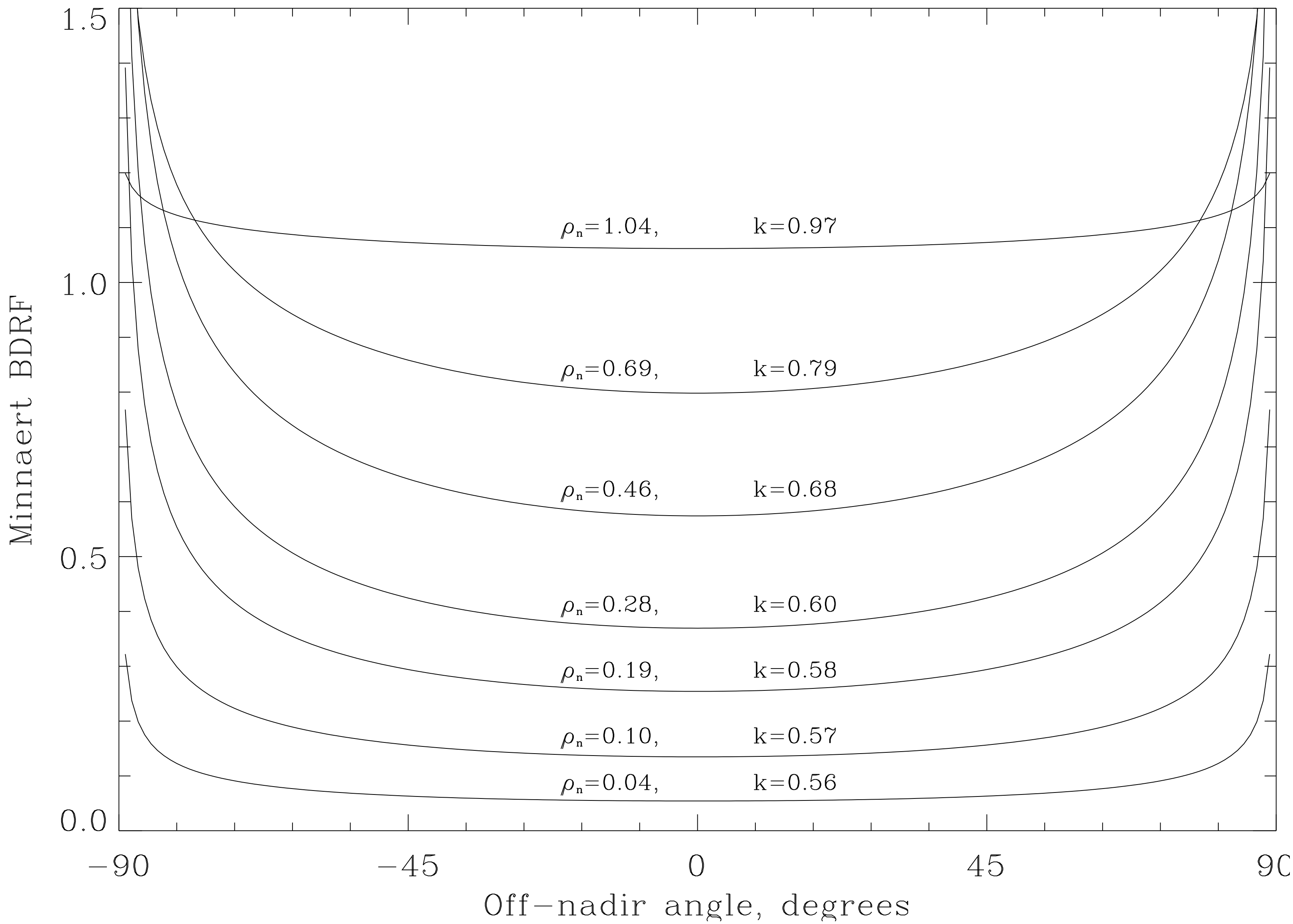


Figure 3: **The BRDF for *Minneart's formula* versus the off-nadir angle. Each curve corresponds to a different surface ranging from very dark to very bright.**

Scattering Geometry

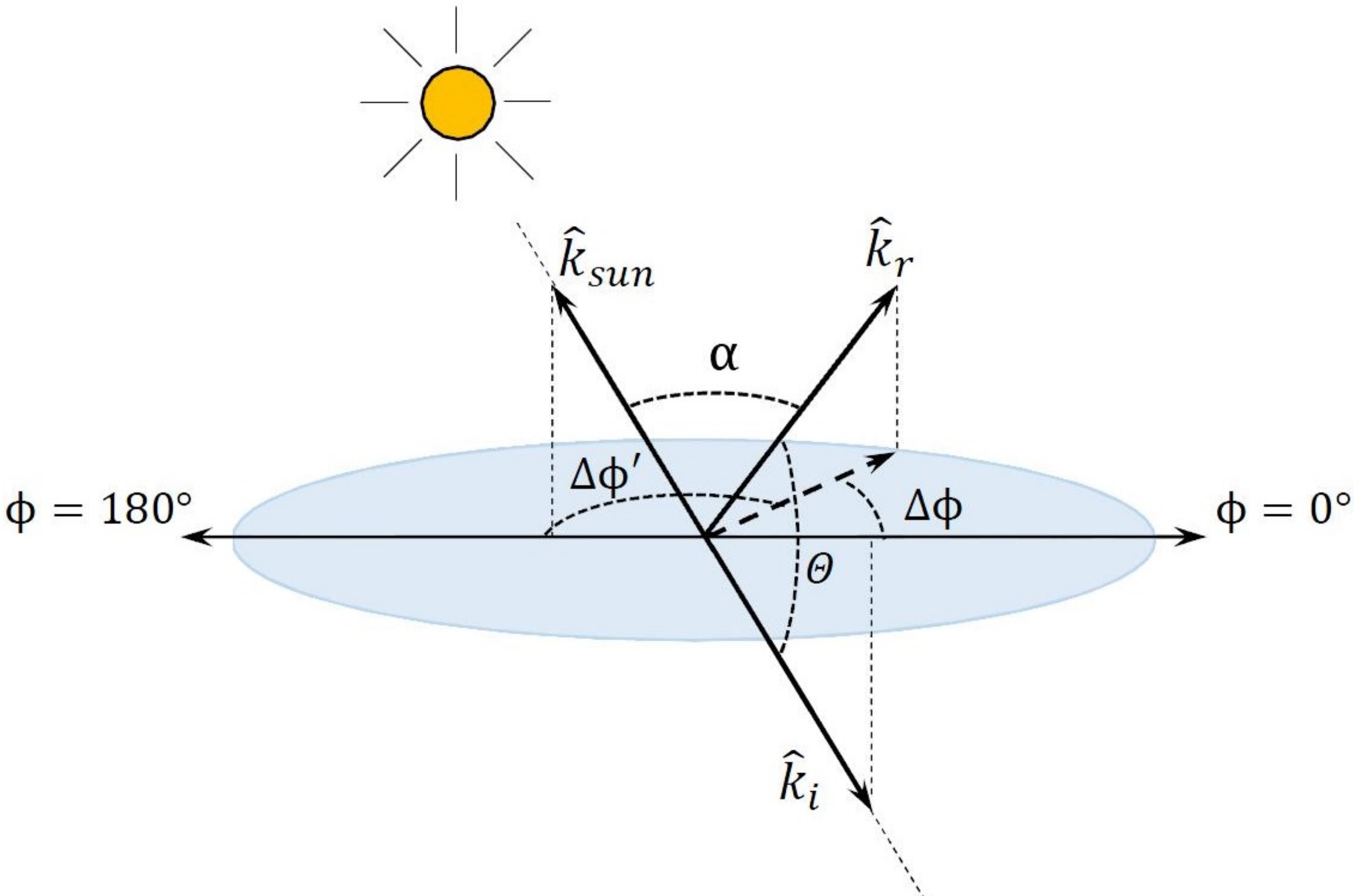


Figure 4: Illustration of the geometry involved in the description of the BRDF. The phase angle α is the supplementary angle of the scattering angle Θ, *i.e.*, $\alpha = \pi - \Theta$. The relative azimuth angle $\Delta\phi = 0^\circ$ corresponds to the glint (specular) direction, and $\Delta\phi = 180^\circ$ to the backscattering (hot-spot) direction.

The Opposition Effect

- The **phase angle** or **backscattering angle** α is defined as the complement of the angle between $\hat{\mathbf{\Omega}}'(\theta', \phi')$ and $\hat{\mathbf{\Omega}}(\theta, \phi)$:

$$\alpha = \pi - \arccos(\hat{\mathbf{\Omega}}' \cdot \hat{\mathbf{\Omega}}) = \pi - \arccos[-\mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos(\phi - \phi')].$$

Planetary surfaces exhibits the opposition effect (O. E.), Heiligenschein, enhanced backscattering or *hot spot* phenomenon (see Fig. 5):

- an abrupt increase in the reflected light as $\alpha \to 0$. Thus, at lunar opposition when the sun is behind the observer, the Moon is brighter than at other phases.

- The opposition effect can be accounted for by a multiplicative correction to the Lommel-Seeliger formula:

$$\rho(\mu_0, \mu, \alpha) = \frac{A}{\mu_0 + \mu}\left\{1 + B\exp[-\frac{1}{h}\tan(\alpha/2)]\right\}.$$

A = normalization constant; $B \geq 0$ is the opposition enhancement factor; h = compaction parameter = measure of angular width of O. E. Correction is maximum at $\alpha = 0$ with a width proportional to h.

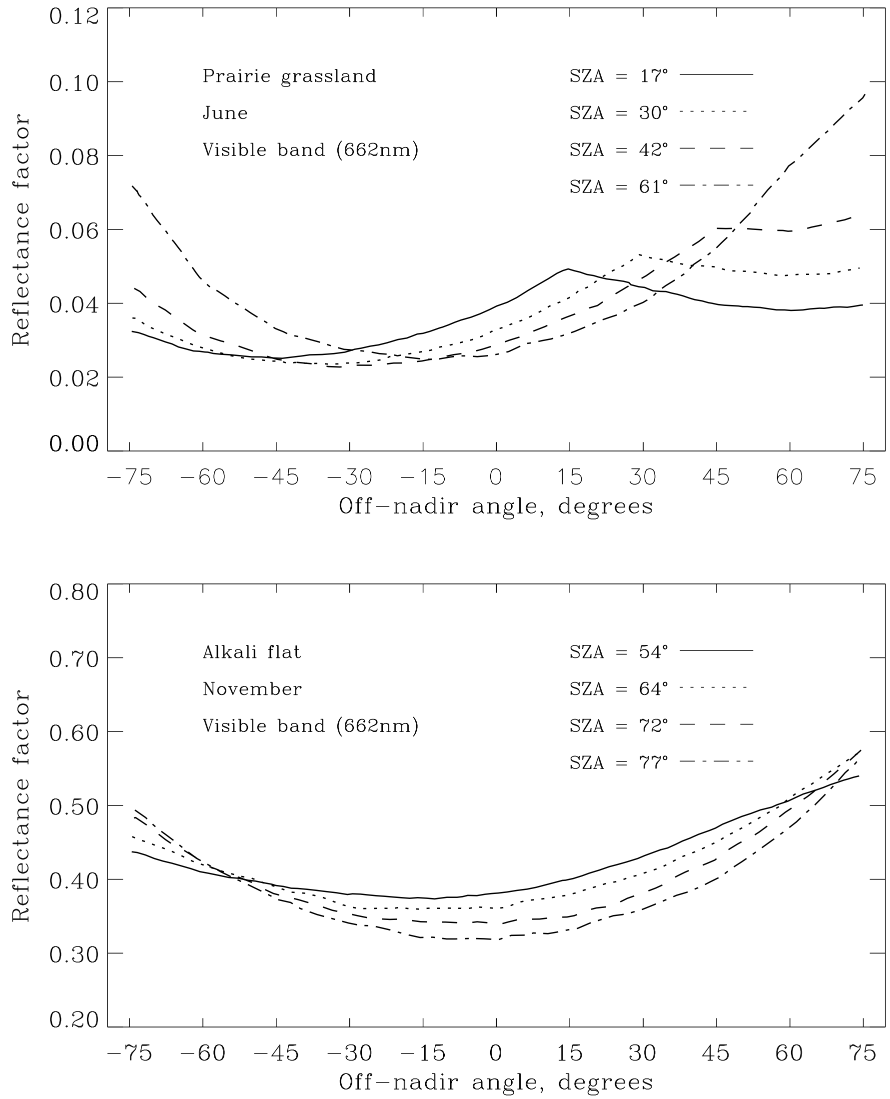


Figure 5: **Measured BRDFs versus the off-nadir angle for prairie grassland (upper panel) and for alkali flat (lower panel).**

Reflection from a wind-roughened sea surface (1)

The slope distribution of surface facets of a wind-roughened body of water can be approximated by a one-dimensional Gaussian distribution derived by Cox and Munk [1954]:

$$
\begin{aligned}
p(\mu_n) &= \frac{1}{\pi\sigma^2}\exp\left(-\frac{1-\mu_n^2}{\sigma^2\mu_n^2}\right) = \frac{1}{\pi\sigma^2}\exp\left(-\frac{\tan^2\theta_n}{\sigma^2}\right) && (4)\\
\mu_n &= \cos\theta_n = \frac{\mu+\mu'}{\sqrt{2(1-\cos\Theta)}} && (5)\\
\sigma^2 &= 0.003 + 0.00512\cdot U && (6)
\end{aligned}
$$

where U is the wind speed in m/s, and the BRDF can be written as:

$$\rho(\mu,\phi;\mu',\phi') = \frac{1}{4\mu'\mu(\mu_n)^4}\cdot p(\mu_n)\cdot r(\cos\Theta, m)\cdot s(\mu,\mu',\sigma). \tag{7}$$

- $r(\cos\Theta, m)$ is the Fresnel reflectance,
- $s(\mu,\mu',\sigma)$ is a shadowing term that is important for large angles of incidence ($\theta_i > 75°$) and
- m is the refractive index of water.

Reflection from a wind-roughened sea surface (2)

$\theta_n = \cos^{-1}\mu_n$ = tilt angle (see Fig. 6): angle between surface normal of a scattering facet for which specular reflection occurs and zenith direction.

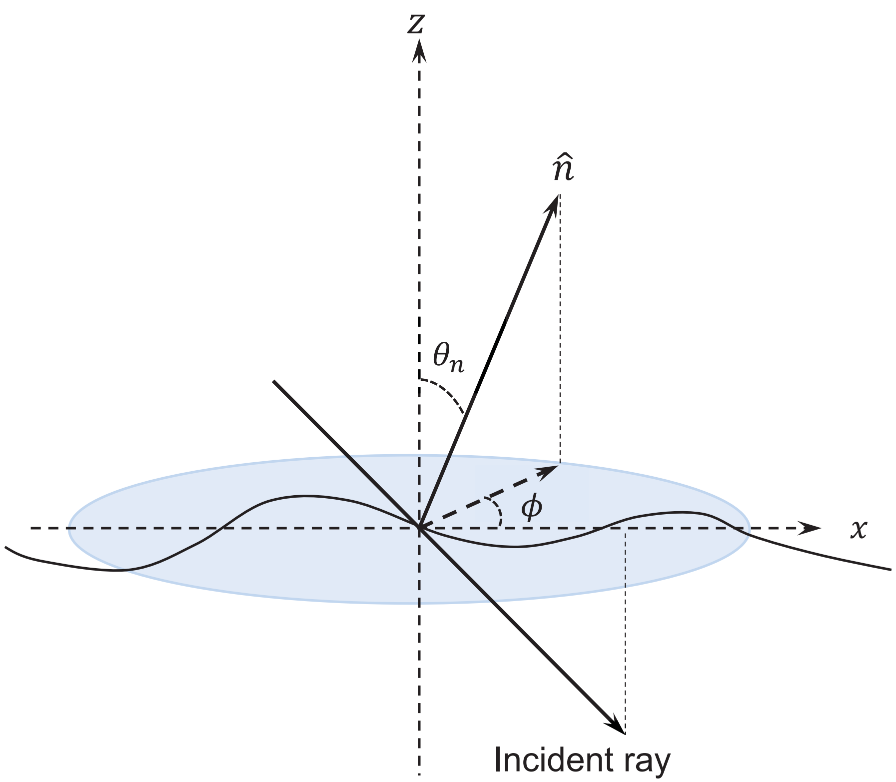


Figure 6: Illustration of tilt angle (θ_n) and relative azimuth angle ($\phi \equiv \Delta\phi$) in rough surface scattering. Here $\hat{\mathbf{n}}$ is the normal to the facet, θ_n is the tilt angle, and ϕ is the azimuth angle relative to the glint direction ($\phi = 0$).

Reflection from a wind-roughened sea surface (3)

Figure 7 shows the Cox-Munk BRDF for several values of the relative azimuth difference, an angle of incidence of $\mu' = \cos 30°$, a refractive index $m = 1.34$ and a wind speed of $U = 5$ m/s.

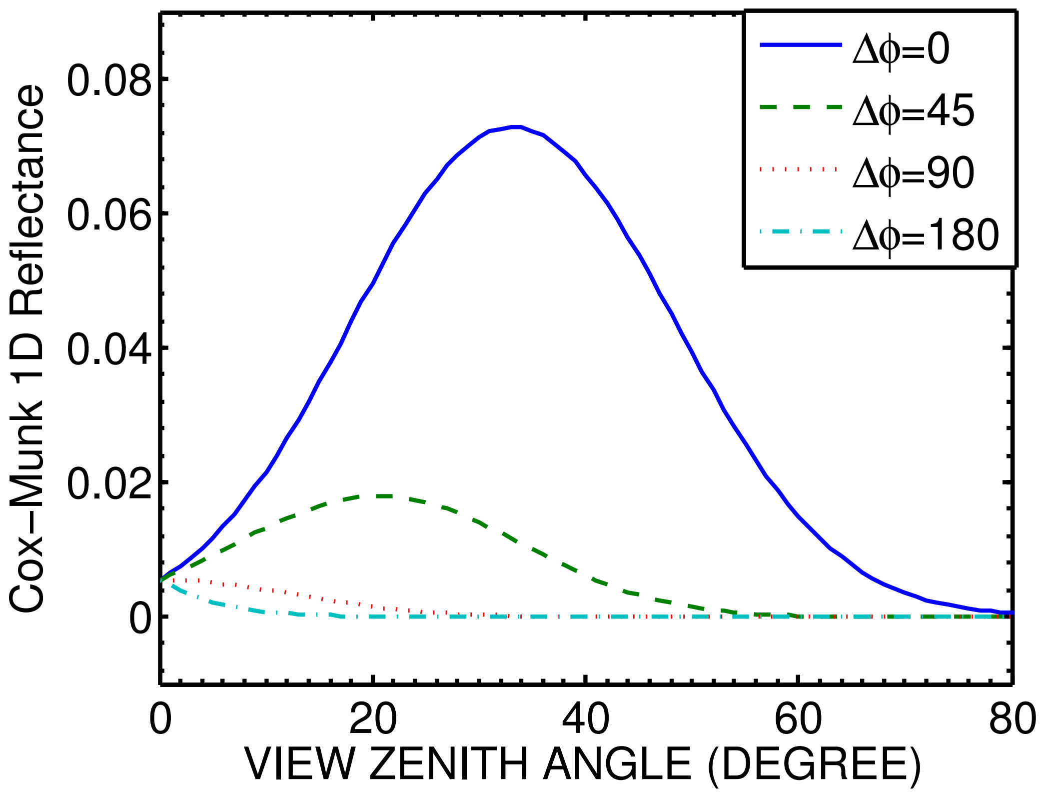


Figure 7: Analytic Cox-Munk 1D Gaussian BRDF for $m = 1.34$, wind speed $U = 5$ m/s, and angle of incidence $= 30°$.

Transmission through a slab medium – (1)

- The transmittance is a measure of how much light reaches the lower surface of a slab either as a direct attenuated beam or as diffuse (scattered) radiation.
- The **spectral bidirectional transmittance or simply transmittance** is defined as the ratio of the transmitted radiance $dI^-_{\nu\mathrm{t}}(\mathbf{\Omega})$ to the incident energy:

$$\mathcal{T}(\nu, -\hat{\mathbf{\Omega}}', -\hat{\mathbf{\Omega}}) \equiv \frac{dI^-_{\nu\mathrm{t}}(\hat{\mathbf{\Omega}})}{I^-_{\nu}(\hat{\mathbf{\Omega}}') \cos\theta' d\omega'}.$$

- Adding up contributions from the beams incident in all downward directions $-\hat{\mathbf{\Omega}}'$, we obtain the total transmitted radiance:

$$I^-_{\nu\mathrm{t}}(\hat{\mathbf{\Omega}}) = \int_- dI^-_{\nu\mathrm{t}}(\hat{\mathbf{\Omega}}) = \int d\omega' \cos\theta' \mathcal{T}(\nu, -\hat{\mathbf{\Omega}}', -\hat{\mathbf{\Omega}}) I^-_{\nu}(\hat{\mathbf{\Omega}}').$$

- Since the transmitted radiance consists of a direct and a diffuse part, we have:

$$I^-_{\nu\mathrm{t}}(\hat{\mathbf{\Omega}}) = I^-_{\nu}(\hat{\mathbf{\Omega}}) e^{-\tau_\mathrm{s}(\nu)} + \int_- d\omega' \cos\theta' \mathcal{T}_\mathrm{d}(\nu, -\hat{\mathbf{\Omega}}', -\hat{\mathbf{\Omega}}) I^-_{\nu}(\hat{\mathbf{\Omega}}')$$

  where $\mathcal{T}_\mathrm{d}(\nu, -\hat{\mathbf{\Omega}}', -\hat{\mathbf{\Omega}})$ = *diffuse transmittance.*

Transmission through a slab medium – (2)

For an incident collimated beam,
$I_\nu^-(\hat{\mathbf{\Omega}}) = F_\nu^{\mathrm{s}}\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_{\mathbf{0}}) = F_\nu^{\mathrm{s}}\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0)$, we get:

$$I_{\nu\mathrm{t}}^-(\hat{\mathbf{\Omega}}) = F_\nu^{\mathrm{s}}\delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0)e^{-\tau_{\mathrm{s}}(\nu)} + F_\nu^{\mathrm{s}}\cos\theta_0\mathcal{T}_{\mathrm{d}}(\nu, -\hat{\mathbf{\Omega}}_0, -\hat{\mathbf{\Omega}}).$$

The corresponding irradiance becomes:

$$F_{\nu\mathrm{t}}^- = \int_- d\omega\cos\theta I_{\nu\mathrm{t}}^-(\hat{\mathbf{\Omega}}) = F_\nu^{\mathrm{s}}\cos\theta_0\big[e^{-\tau_{\mathrm{s}}(\nu)} + \int_- d\omega\cos\theta\;\mathcal{T}_{\mathrm{d}}(\nu, -\hat{\mathbf{\Omega}}_0, \hat{\mathbf{\Omega}})\big].$$

- The **transmittance** is the ratio of the transmitted irradiance to the collimated incident irradiance ($e^{-\tau_{\mathrm{s}}(\nu)}$ is the *beam transmittance* $\mathcal{T}_{\mathrm{b}}(\nu)$):

$$\mathcal{T}(\nu, -\hat{\mathbf{\Omega}}_0, -2\pi) \equiv \frac{F_{\nu\mathrm{t}}^-}{F_\nu^{\mathrm{s}}\cos\theta_0} = e^{-\tau_{\mathrm{s}}(\nu)} + \int_- d\omega\cos\theta\;\mathcal{T}_{\mathrm{d}}(\nu, -\hat{\mathbf{\Omega}}_0, -\hat{\mathbf{\Omega}}).$$

- For a uniform incident radiation field $\mathcal{I}_\nu$, the transmittance is:

$$\mathcal{T}_{\mathrm{tot}}(\nu) = \frac{\int_{2\pi} d\omega\cos\theta\;\mathcal{I}_\nu\;e^{-\tau(\nu)/\mu}}{\pi\mathcal{I}_\nu} = 2\int_0^1 d\mu\mu\;e^{-\tau(\nu)/\mu} = 2E_3(\tau(\nu)),$$

where $E_3(\tau(\nu))$ is the exponential integral of order 3.

Spherical or Bond Albedo

- For definition of the spherical or bond albedo we refer to Fig. 8:
- Area of annulus presented to the Sun: $dA = 2\pi R \cdot R \sin\theta_0 \cdot \cos\theta_0 d\theta_0$
- Solar energy received by annulus:

$$dAF_\nu^{\mathrm{s}} = F_\nu^{\mathrm{s}}[2\pi R^2 \sin\theta_0 \cos\theta_0 d\theta_0] = 2\pi R^2 F_\nu^{\mathrm{s}} \mu_0 d\mu_0.$$

- Energy reflected from annulus:

$$\rho(\nu, \mu_0, 2\pi)(dAF_\nu^{\mathrm{s}}) = 2\pi R^2 F_\nu^{\mathrm{s}} \rho(\nu, -\mu_0, 2\pi)\mu_0 d\mu_0.$$

- Integration over the entire planetary disk gives the total spectral reflected energy.
- Total incoming solar energy: $\pi R^2 F_\nu^{\mathrm{s}}$.
- The spectral **spherical** albedo (**bond** albedo) is the ratio of the disk-integrated reflected energy to the disk-integrated incoming solar energy:

$$\bar{\rho}(\nu) = \frac{2\pi R^2 F_\nu^{\mathrm{s}} \int_0^1 \rho(\nu, -\mu_0, 2\pi)\mu_0 d\mu_0}{\pi R^2 F_\nu^{\mathrm{s}}} = 2\int_0^1 d\mu_0 \mu_0 \rho(\nu, -\mu_0, 2\pi).$$

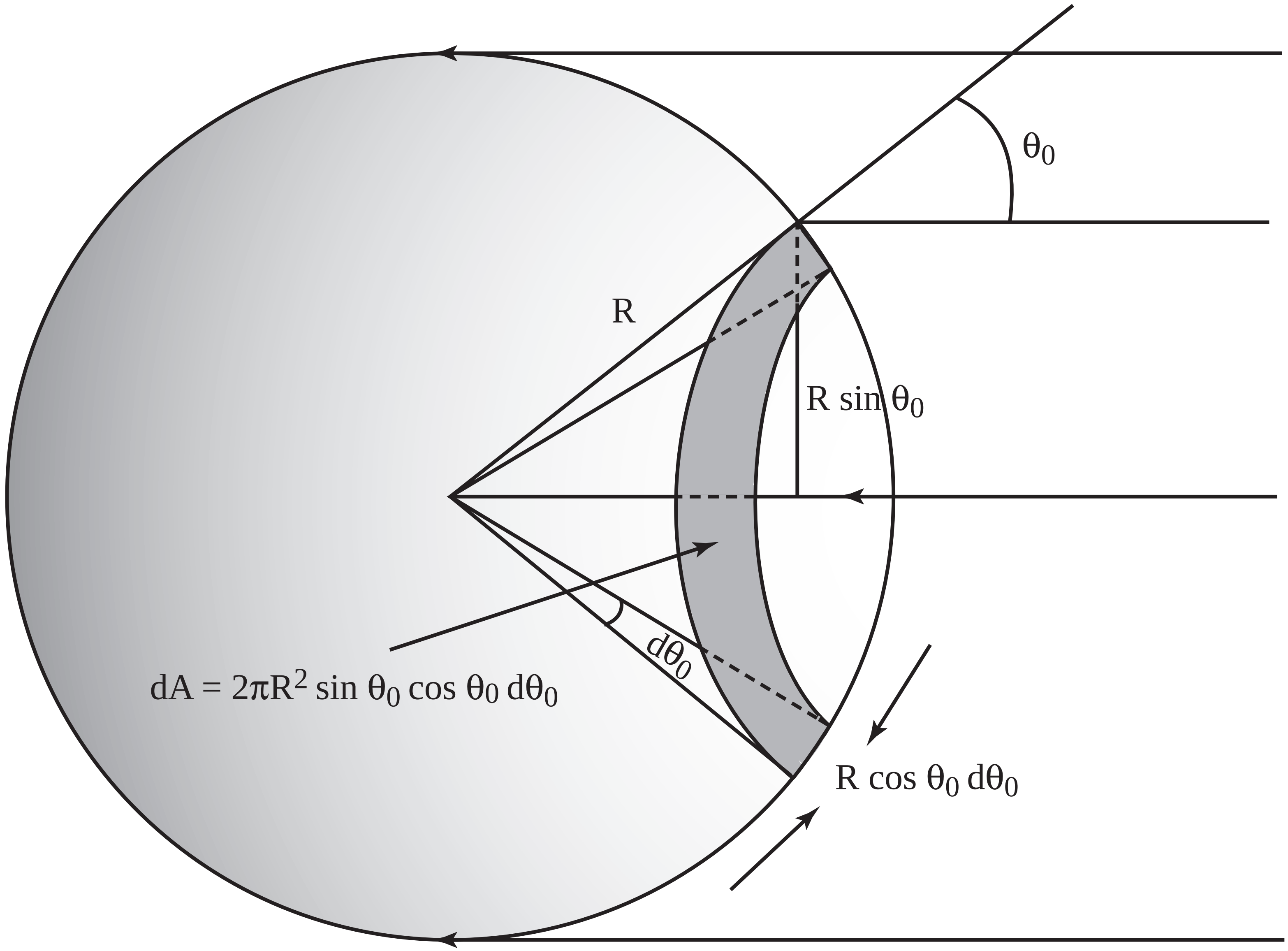


Figure 8: **Geometry for the definition of the spherical albedo.**

Spherical Transmittance and Absorptance

- Spectral spherical transmittance and absorptance are defined analogously:

$$\bar{\mathcal{T}}(\nu) = 2\int_0^1 \mathcal{T}(\nu, -\mu_0, 2\pi)\mu_0 d\mu_0$$

$$\bar{\alpha}(\nu) = 2\int_0^1 \alpha(\nu, -\mu_0, 2\pi)\mu_0 d\mu_0$$

- Frequency - integrated quantities:

$$\bar{\rho} = \int_0^\infty \bar{\rho}(\nu)d\nu$$
$$\bar{\alpha} = \int_0^\infty \bar{\alpha}(\nu)d\nu$$
$$\bar{\mathcal{T}} = \int_0^\infty \bar{\tau}(\nu)d\nu.$$

Kirchhoff's Law: Volume Absorption and Emission (1)

To discuss radiative processes in an extended medium we must define absorption and emission per unit volume. For a **hohlraum** filled with matter throughout the volume, which both scatters and absorbs radiation at frequency ν:

- **Kirchhoff's Law** relates the **thermal emission coefficient** j_ν^{th} to the absorption coefficient and the Planck function

Kirchhoff's Law Applied to Volume Emission

$$j_\nu^{th} = \alpha(\nu) B_\nu(T). \tag{8}$$

- Since $S_\nu \equiv j_\nu / k(\nu)$ (Eq. 2.27), we define the thermal contribution to the source function by:

$$S_\nu^{th} \equiv \frac{j^{th}}{k(\nu)} = \frac{\alpha(\nu)}{k(\nu)} B_\nu(T). \tag{9}$$

Kirchhoff's Law: Volume Absorption and Emission (2)

- We define the spectral **volume emittance** $\epsilon_v(\nu, \hat{\Omega}, T)$, as the ratio of the thermal emission per unit volume of the matter under consideration to that of a perfectly "black" material of the same mass and temperature T, $S_\nu^{\mathrm{BB}} \equiv B_\nu(T)$:

$$\epsilon_v(\nu, \hat{\Omega}, T) = \frac{S_\nu^{th}(\hat{\Omega}, T)}{B_\nu(T)}.$$

Most atmospheric and oceanic absorbers are isotropic emitters, so that the absorption coefficient is independent of angle.

- Thus, dropping the angular dependence and using Eq. (9), we find:

$$\epsilon_v(\nu, T) = \frac{\alpha(\nu, T)}{k(\nu)}. \tag{10}$$

- **The volume emittance ϵ_v is proportional to the absorption coefficient, α.**

Kirchhoff's Law: Volume Absorption and Emission (3)

A planetary medium is far from an artificial closed system such as a **hohlraum**. It is therefore surprising that within spectral lines in the IR, planetary media radiate approximately as a blackbody:

- **the source function is equal to the Planck function.**

This situation prevails when (recall discussion for two-level atom):

- the collisional rates of excitation/de-excitation of the quantum states are much larger than the corresponding radiative loss rates.
- Then the populations of these states are determined by the local kinetic temperature of the medium, rather than by the radiation field.
- In the troposphere thermal IR radiation is in **local thermodynamic equilibrium** (LTE), which implies that Eqs. 8–10 are valid, but the radiance is not equal to the Planck function, as it would be in TE. (At very low atmospheric densities LTE will break down even in the IR, and Eqs. 8–10 are no longer valid.)

Kirchhoff's Law: Volume Absorption and Emission (4)

- However, LTE does **not** apply to shortwave radiative processes, because
- the kinetic energy involved in thermal collisions ($\sim k_{\rm B}T$) is much less than the excitation energies associated with visible and UV transitions (E_i).
- Thus, at shortwave energies the collisional quenching rates greatly exceed the collisional excitation rates.
- Absorption of sunlight, followed by collisional quenching, heats the medium. The absence of the opposite process (collisional energy converting to radiative energy) "uncouples" the shortwave radiation from the thermal state of the gas.
- There is little overlap between the radiation spectra of the Sun and the Earth (Fig. 9): We may treat the two spectra separately.

Due to lack of strong absorption by the major atmospheric gases in the visible spectrum:

- The major shortwave interaction is in the UV spectrum below 300 nm where sunlight never reaches the surface.

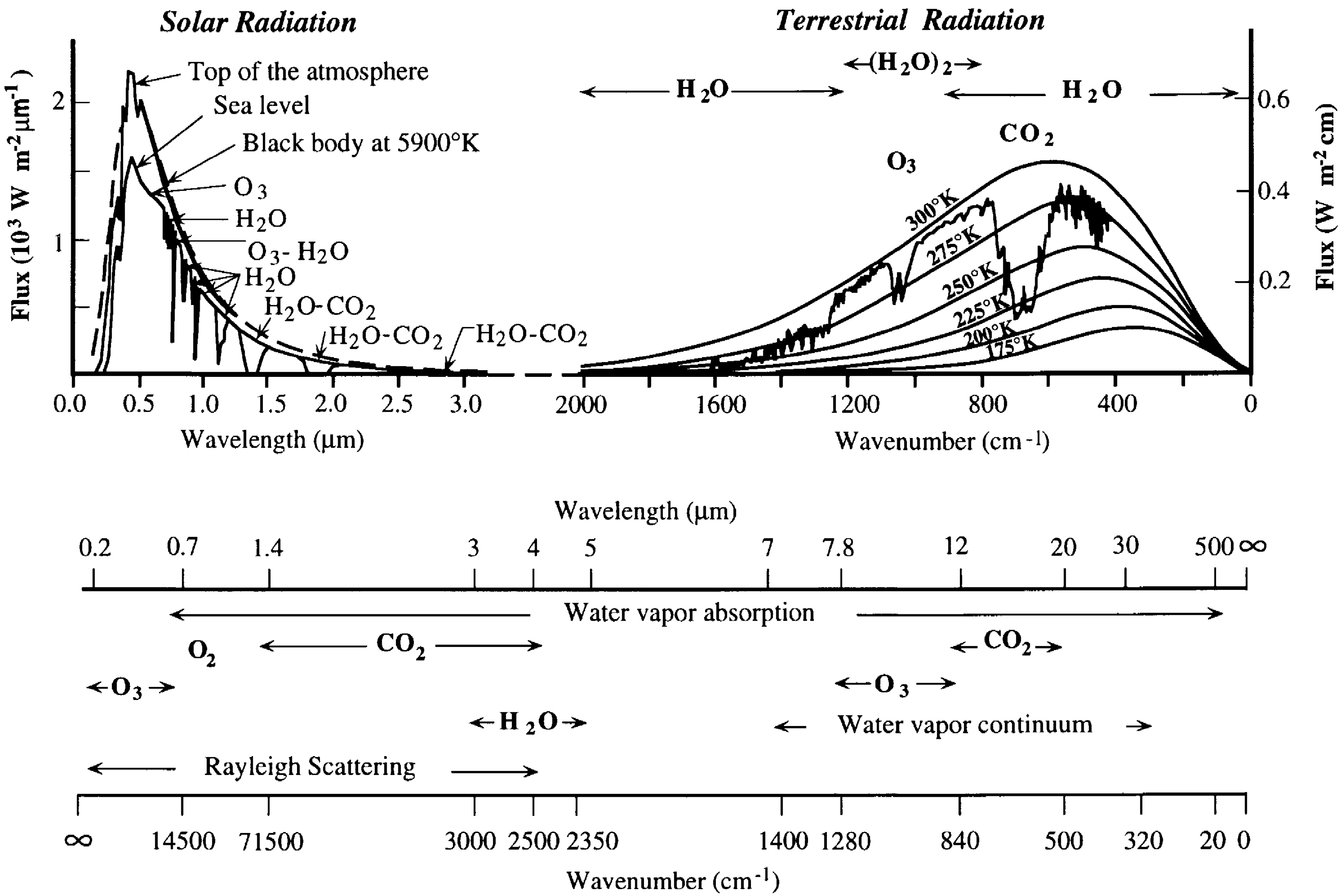


Figure 9: Spectral distribution of solar and terrestrial radiation fields. Also shown are the approximate shapes and positions of the scattering and absorption features of the Earth's atmosphere.

Differential Equation of Radiative Transfer (1)

- Recall that: the radiative energy that is incident normally on the area dA in the direction $\hat{\Omega}'$ and within the solid angle $d\omega'$ centered around $\hat{\Omega}'$, in time dt and within frequency interval $d\nu$ is the fourth-order quantity (see Fig. 10):

$$d^4E' = I_\nu(\hat{\Omega}')\, dA\, dt\, d\nu d\omega'.$$

- The radiative energy which is scattered in **all** directions is:

$$\sigma\ ds\ d^4E' = \sigma\ ds\ I_\nu(\hat{\Omega}')\, dA\, dt\, d\nu d\omega',$$

where ds is the length of the scattering volume element in the direction normal to dA, and σ is the scattering coefficient.

- We are interested in **that fraction of the scattered energy which is directed into the solid angle $d\omega$ centered around the direction $\hat{\Omega}$.**

- This fraction is proportional to $p(\hat{\Omega}', \hat{\Omega})\, d\omega/4\pi$, where $p(\hat{\Omega}', \hat{\Omega})$ is the scattering phase function (see §3.4).

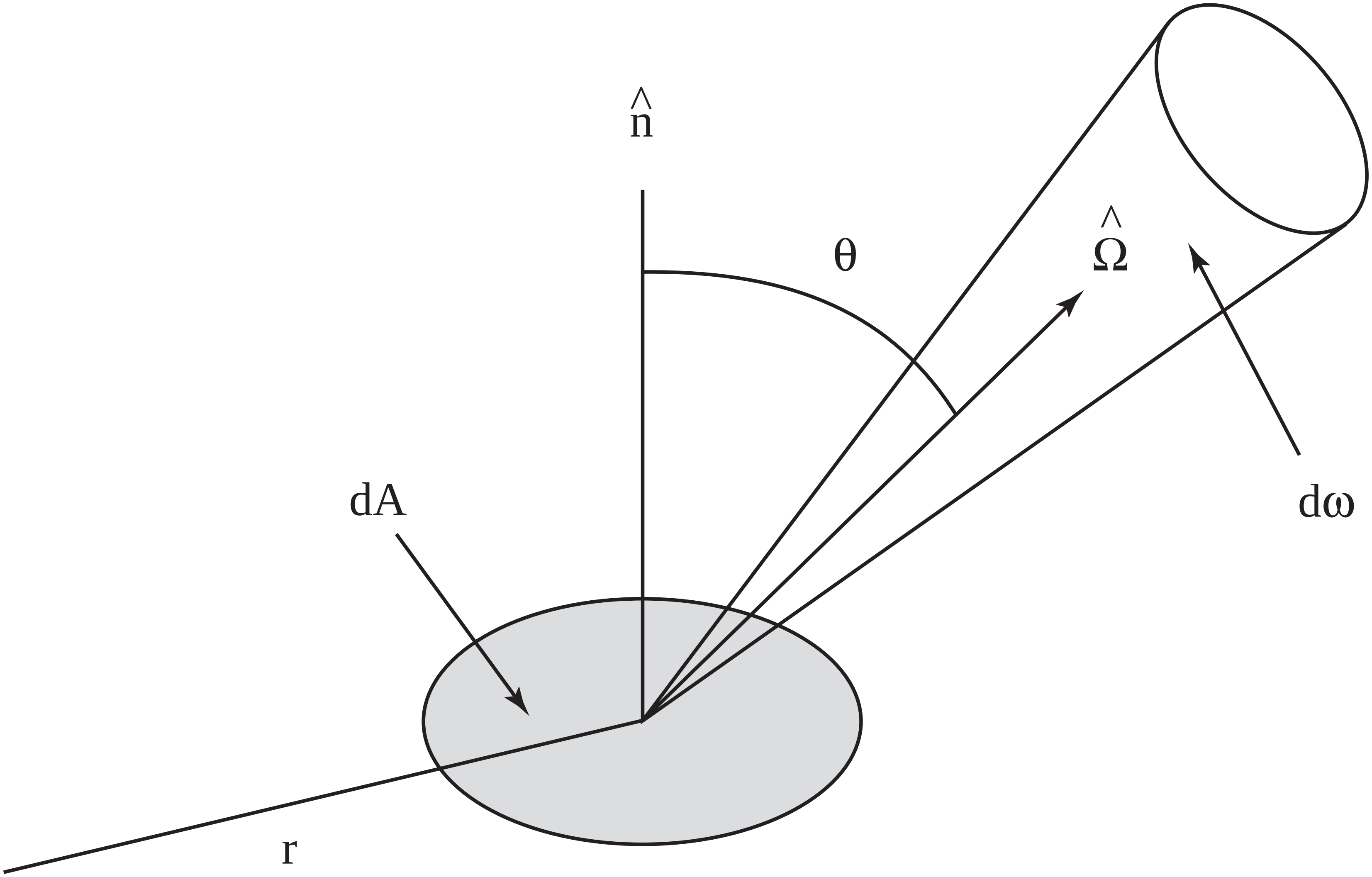


Figure 10: **The flow of radiative energy carried by a beam in the direction $\hat{\Omega}$ through a transparent surface element dA. The flow direction $\hat{\Omega}$ is at an angle θ with respect to the surface normal $\hat{n}$ ($\cos\theta = \hat{n} \cdot \hat{\Omega}$).**

Differential Equation of Radiative Transfer (2)

If we multiply the scattered energy by this fraction, *i.e.* form

$$\sigma ds d^4E' \; p(\hat{\Omega}', \hat{\Omega}) \; d\omega/4\pi = \sigma ds I_\nu(\hat{\Omega}') dA dt d\nu d\omega' \; p(\hat{\Omega}', \hat{\Omega}) d\omega/4\pi,$$

and then integrate over all incoming directions $d\omega'$:

- we find that the total scattered energy emerging from the volume element $dV = ds\, dA$ in the direction $\hat{\Omega}$ is:

$$d^4E = \sigma(\nu) dV \, dt \, d\nu \, d\omega \int_{4\pi} d\omega' \frac{p(\hat{\Omega}', \hat{\Omega})}{4\pi} I_\nu(\hat{\Omega}'). \tag{11}$$

- We define the **emission coefficient for scattering** as:

$$j_\nu^{sc} \equiv \frac{d^4E}{dV \, dt \, d\nu \, d\omega} = \sigma(\nu) \int_{4\pi} \frac{d\omega'}{4\pi} p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}').$$

- The source function for scattering is thus:

$$S_\nu^{sc}(\hat{\Omega}) = \frac{j_\nu^{sc}}{k(\nu)} = \frac{\sigma(\nu)}{k(\nu)} \int_{4\pi} \frac{d\omega'}{4\pi} p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}'). \tag{12}$$

Differential Equation of Radiative Transfer (3)

The quantity $\sigma(\nu)/k(\nu)$ appearing in Eq. 12 is called the **single-scattering albedo**, $\varpi(\nu)$. Since $k(\nu) = \sigma(\nu) + \alpha(\nu)$, it is clear that

$$\varpi(\nu) = \sigma(\nu)/[\sigma(\nu) + \alpha(\nu)] \leq 1.$$

- We interpret $\varpi(\nu)$ as the **probability that a photon will be scattered, given an extinction event**.
- Given that an interaction has occurred, the quantity $1-\varpi(\nu)$ is the **probability of absorption per extinction event** or the **co-albedo**.
- For thermal emission, $\epsilon_v = \alpha(\nu)/k(\nu) = 1 - \varpi(\nu)$ is the **volume emittance**. Thus, the complete time-independent radiative transfer equation is:

Radiative Transfer Equation including Absorption Thermal Emission and Multiple Scattering

$$\frac{dI_\nu(\hat{\Omega})}{d\tau_{\rm s}} = -I_\nu + \underbrace{[1 - \varpi(\nu)]B_\nu(T)}_{\text{thermal emission}} + \underbrace{\frac{\varpi(\nu)}{4\pi}\int_{4\pi} d\omega'\, p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}')}_{\text{multiple scattering}}. \qquad (13)$$

Solution of the Radiative Transfer equation for zero scattering (1)

Equation 13 includes both scattering (Eq. 12) and thermal emission (Eq. 9):

- It reveals the major mathematical complexity of radiative transfer theory: it involves the solution of an **integro-differential equation**.
- In the limit of no scattering ($\varpi(\nu) = 0$), the radiation is affected only by absorption and emission processes. Then Eq. 13 simplifies to:

$$\frac{dI_\nu}{d\tau_{\mathrm{s}}} = -I_\nu + B_\nu(T) \tag{14}$$

where the source function $S_\nu = B_\nu(T)$ may be considered to be a known.

- This **local** problem is much easier to solve than the more general **non-local** problem involving multiple scattering.
- Note that the slant optical depth τ_{s} is measured along the beam direction, taken to be a straight line since we are ignoring refraction.

Solution of the Radiative Transfer equation for zero scattering (2)

- A solution of Eq. 14, satisfying the appropriate boundary conditions, yields the radiation field I_ν at all positions τ_s along the beam direction.
- The solution will clearly vary with the frequency ν, the temperature T, and the optical properties of the medium, embodied in the absorption coefficient $\alpha(\nu)$, which in general may vary from point to point in the medium.
- We consider an inhomogeneous medium in LTE, which at each point radiates thermal emission according to the Planck function at the local temperature.
- The medium may have arbitrarily shaped boundaries. It is illuminated by a beam of radiation in the direction $\hat{\Omega}$ at the boundary point P_1 (see Fig. 11).
- We want to find the **elementary solution** for the radiance $I_\nu(P_2, \hat{\Omega})$ which emerges from the medium at point P_2 along the same direction.

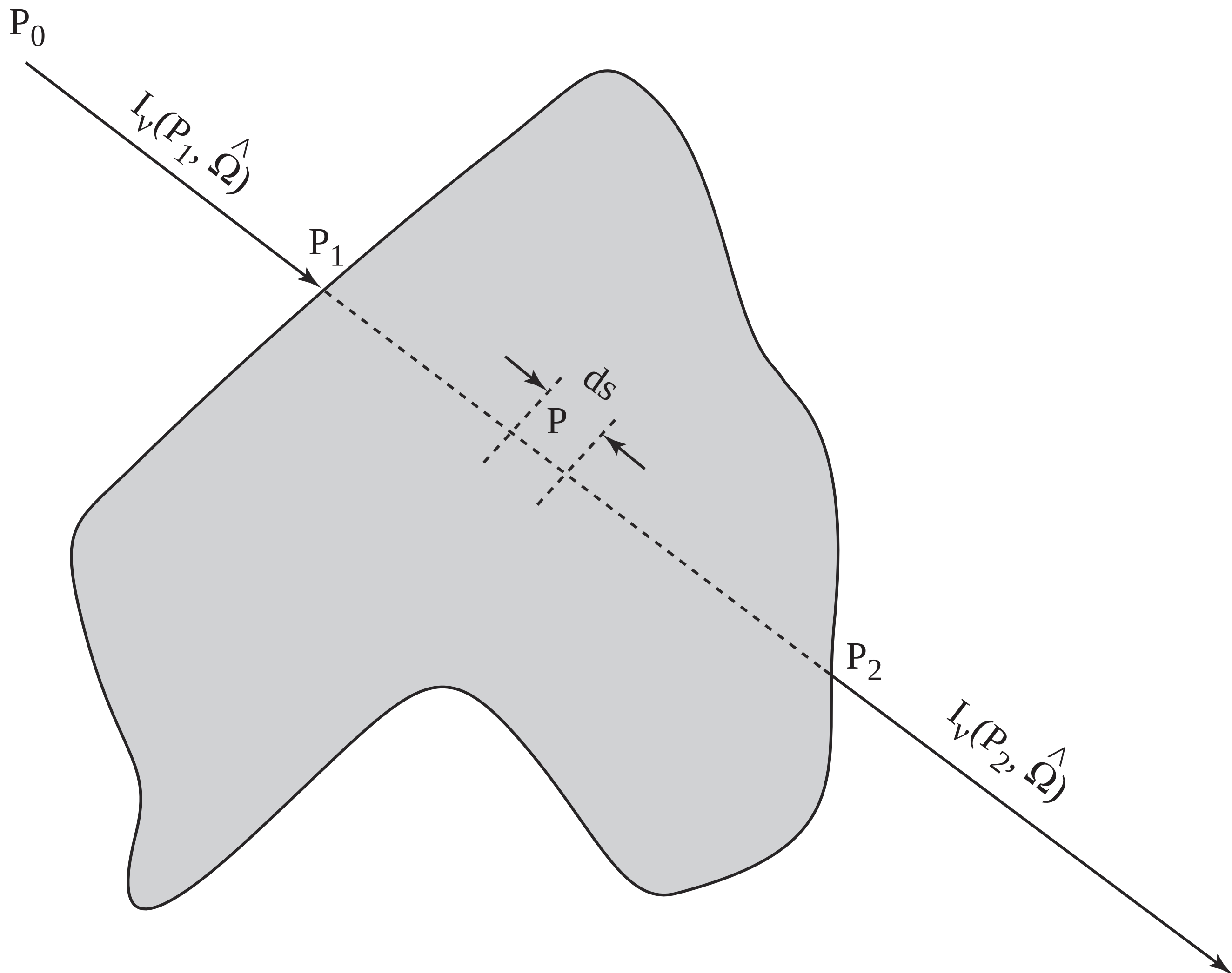


Figure 11: **A beam of radiation is incident on an absorbing/emitting region at the boundary point P_1. It is attenuated along the path P_1P_2, and emerges at the point P_2. The propagation direction of the beam is denoted by $\hat{\Omega}$. In addition, thermal emission adds to the beam at all points within the medium.**

Solution of the Radiative Transfer equation for zero scattering (3)

- A completely general distribution of radiance in angle and frequency can be obtained by repeating the elementary solution $I_\nu(P_2, \hat{\Omega})$ for all incident beams and for all frequencies.
- The elementary solution will be found to be a sum of two terms:
  (a) the incident radiance $I_\nu(P_1, \hat{\Omega})$ attenuated by the intervening optical path along P_1P_2, and
  (b) the contributions from the internal sources at all points P between P_1 to P_2 and attenuated by the intervening optical path along PP_2.
- Eq. (14) is readily integrated by using an **integrating factor**, which in this case is e^{τ_s}. After multiplying by e^{τ_s}, Eq. 14 may be written as a perfect differential:

$$\frac{dI}{d\tau_s} e^{\tau_s} + I e^{\tau_s} = \frac{d}{d\tau_s}\left(I e^{\tau_s}\right) = B e^{\tau_s}. \tag{15}$$

Solution of the Radiative Transfer equation for zero scattering (4)

Ignoring refraction, we integrate along a straight path from the point P_1 to the point P_2, the latter point being at the boundary of the medium (see Fig. 11).

- The optical path from the point P_1 to an intermediate point P is given by:

$$\tau_{\rm s}(P_1, P) = \int_{P_1}^{P} \alpha \, ds = \int_0^P \alpha \, ds - \int_0^{P_1} \alpha \, ds \equiv \tau_{\rm s}(P) - \tau_{\rm s}(P_1).$$

- The optical path may be measured from an arbitrary reference point P_0 outside the medium, even though $\tau(P_0, P_1) = 0$. $\tau_{\rm s}$ increases monotonically with distance in the medium due to the addition of absorbing matter along the beam.

- Integration of Eq. (15) along the straight line from P_1 to P_2 yields:

$$\int_{\tau_{\rm s}(P_1)}^{\tau_{\rm s}(P_2)} d\tau_{\rm s}' \frac{d}{d\tau_{\rm s}'}\left(I e^{\tau_{\rm s}'}\right) = I[\tau_{\rm s}(P_2)]e^{\tau_{\rm s}(P_2)} - I[\tau_{\rm s}(P_1)]e^{\tau_{\rm s}(P_1)} = \int_{\tau_{\rm s}(P_1)}^{\tau_{\rm s}(P_2)} d\tau_{\rm s}' B(\tau_{\rm s}') e^{\tau_{\rm s}'}$$

where the integration variable $\tau_{\rm s}'$ stands for the slant optical depth.

Solution of the Radiative Transfer equation for zero scattering (5)

- Solving for the radiance at the point P_2, we find:

$$\begin{aligned} I[\tau_{\rm s}(P_2)] &= I[\tau_{\rm s}(P_1)]e^{-\tau_{\rm s}(P_2)+\tau_{\rm s}(P_1)} + \int_{\tau_{\rm s}(P_1)}^{\tau_{\rm s}(P_2)} d\tau_{\rm s}' B(\tau_{\rm s}') e^{\tau_{\rm s}'-\tau_{\rm s}(P_2)} \\ &= I[\tau_{\rm s}(P_1)]e^{-\tau_{\rm s}(P_1,P_2)} + \int_{\tau_{\rm s}(P_1)}^{\tau_{\rm s}(P_2)} d\tau_{\rm s}' B(\tau_{\rm s}') e^{-\tau_{\rm s}'(P,P_2)}. \end{aligned} \tag{16}$$

This result has a direct physical interpretation. The radiance at the point P_2 emerging from the medium in the beam direction P_1P_2 consists of two parts:

1. The first term is the contribution from the radiance incident at the boundary point P_1, which has been attenuated by the beam transmittance.
2. The second term is due to thermal emission from the medium which lies along the beam, weighted by the appropriate transmittance $e^{-\tau_{\rm s}(P,P_2)}$. Note:

- The above solution is valid whether or not the points P_1 and P_2 lie at the boundaries of the medium (both P_1 and P_2 may be interior points).

Solution with Zero Scattering in Slab Geometry (1)

- The most common geometry in the theory of radiative transfer is that of a **plane-parallel** medium, or a **slab** (see Fig. 12). This geometry is appropriate because:

- gravity imposes a density stratification, so that the medium properties tend to vary primarily in the vertical direction. In many cases, we can ignore the horizontal variation in the medium.

- We will distinguish the vertical optical path τ (which we hereafter call the **optical depth**) from the slant optical depth τ_{s}.

- It is convenient to measure the optical depth along the vertical direction downward from the 'top' of the medium.

The relationship between the vertical and the slant optical depths is:

$$\tau(z) \equiv \int_z^{\infty} dz' \, k(z') = \tau_{\mathrm{s}} |\cos\theta| = \tau_{\mathrm{s}} |u|$$

where $\theta = \cos^{-1} u$ is the polar angle of the beam direction.

Solution with Zero Scattering in Slab Geometry (2)

- Since it is used so frequently, we assign a special symbol, u to $\cos\theta$, so that $d\tau_{\rm s} = -k\,dz/|u| = -k\,dz/\mu$, $\mu = |\cos\theta| = |u|$.

- The extinction optical depth can also be written in terms of the vertical column number $\mathcal{N}$ and the extinction cross section k_n (see Eqs. 1.10 and 2.20)

$$\tau(z) = k_n \int_z^\infty dz'\, n(z') \equiv k_n \mathcal{N}(z)$$

or in terms of the extinction coefficient k_m (see Eq. 2.19):

-

$$\tau(z) = k_m \int_z^\infty dz' \rho(z') \equiv k_m \mathcal{M}(z),$$

where $\mathcal{M}(z)$ is the mass of the material in a vertical column of unit cross-sectional area.

- If R is the radial distance from the center of the planet, and H is the vertical scale length of the absorber, the slab approximation is valid if $H/R \ll 1$, and θ not too close to $90°$.

- If these conditions are violated, it is necessary to take into account the curvature of the atmospheric layers.

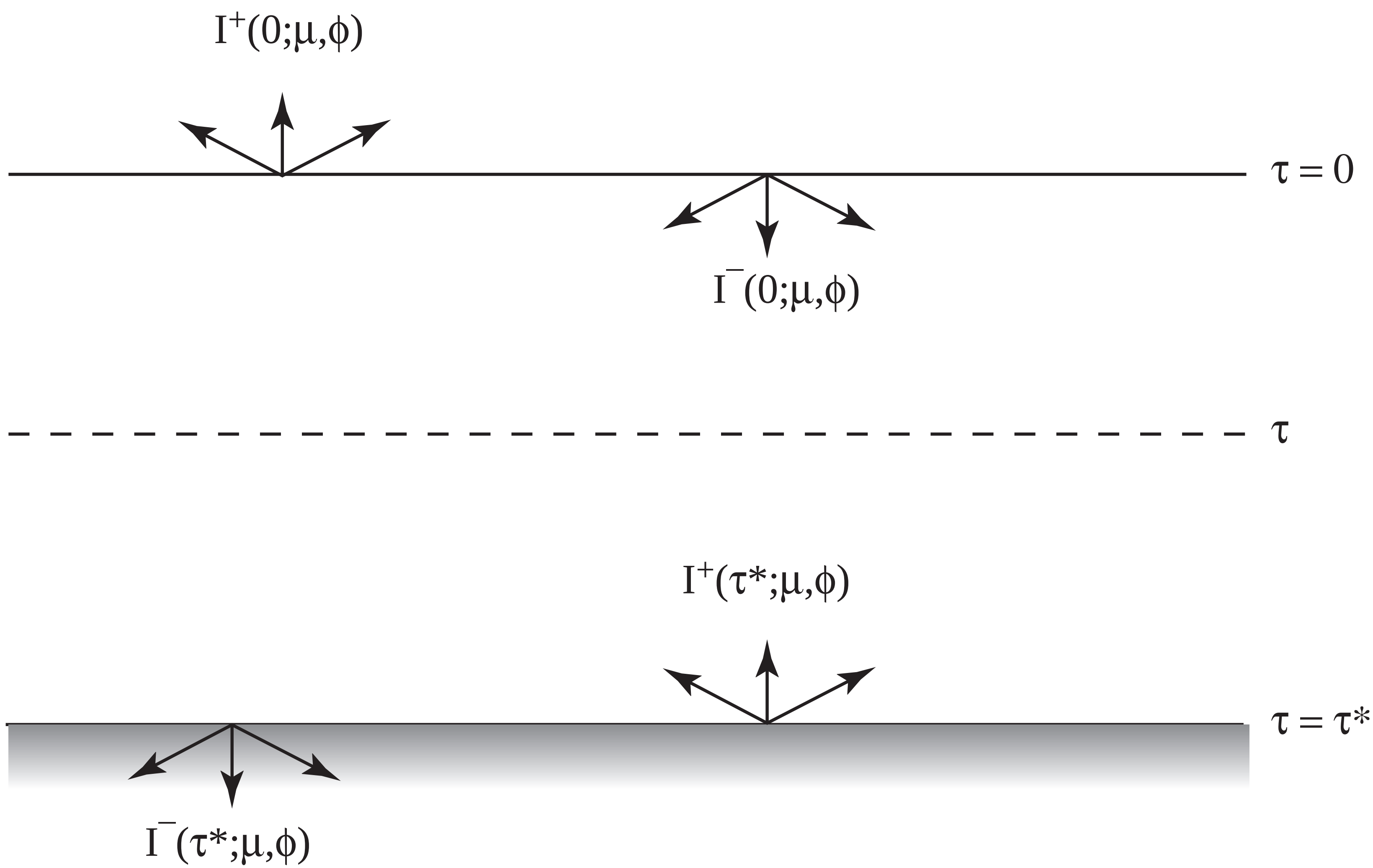


Figure 12: **Half-range radiances in a slab geometry. The optical depth variable τ is measured downward from the 'top' of the medium ($\tau = 0$) to the 'bottom' ($\tau = \tau^*$). $\mu = |u| = |\cos\theta|$ is equal to the absolute value of the cosine of the angle θ, the polar angle of the propagation vector $\hat{\Omega}$.**

Half-range Quantities in a Slab Geometry (1)

- The **half-range radiances** are defined by (see Fig. 12):

$$I_\nu^+(\tau,\theta,\phi) \equiv I_\nu(\tau,\theta \le \pi/2,\phi)$$
$$I_\nu^-(\tau,\theta,\phi) \equiv I_\nu(\tau,\theta > \pi/2,\phi). \tag{17}$$

These definitions may also be expressed in terms of $u = \cos\theta \ge 0$, and $u < 0$.

- It will become apparent later that the variable $\mu = |u|$ makes the notation for slant optical depth simple and straightforward.

- The irradiance defined in terms of half-range quantities:

$$\begin{aligned} F_\nu^+(\tau) &= \int_+ d\omega \cos\theta I_\nu^+(\hat{\Omega}) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta I_\nu^+(\tau,\theta,\phi) \\ &= \int_0^{2\pi} d\phi \int_0^1 d\mu \; \mu I_\nu^+(\tau,\mu,\phi) \end{aligned} \tag{18}$$

$$\begin{aligned} F_\nu^-(\tau) &= -\int_- d\omega \cos\theta I_\nu^-(\hat{\Omega}) = -\int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta I_\nu^-(\tau,\theta,\phi) \\ &= \int_0^{2\pi} d\phi \int_0^1 d\mu \; \mu I_\nu^-(\tau,\mu,\phi). \end{aligned} \tag{19}$$

Half-range Quantities in a Slab Geometry (2)

- The downward irradiance F^- is seen to be a positive quantity. The net irradiance is:

$$
\begin{aligned}
F_\nu(\tau) &= \int_{4\pi} d\omega \cos\theta I_\nu(\hat{\Omega}) = \int_+ d\omega \cos\theta I_\nu^+(\hat{\Omega}) + \int_- d\omega \cos\theta I_\nu^-(\hat{\Omega}) \\
&= F_\nu^+(\tau) - F_\nu^-(\tau).
\end{aligned} \tag{20}
$$

- Note: the net irradiance in slab geometry is positive if the net radiative energy flows in the upward (positive) direction, or toward increasing z and decreasing τ.

In the limit of no scattering the radiative transfer equations for the half-range radiances become:

$$\mu \frac{dI_\nu^+(\tau,\mu,\phi)}{d\tau} = I_\nu^+(\tau,\mu,\phi) - B_\nu(\tau) \tag{21}$$

$$-\mu \frac{dI_\nu^-(\tau,\mu,\phi)}{d\tau} = I_\nu^-(\tau,\mu,\phi) - B_\nu(\tau). \tag{22}$$

- Note: the independent variable is now the absorption optical depth, measured downwards from the 'top' of the medium, which accounts for the difference in sign of the LHS of Eqs. 21 and 22.

Formal Solution in a Slab Geometry (1)

- We first obtain a formal solution of Eq. 22. Choosing the integrating factor to be $e^{\tau/\mu}$, we obtain:

$$\frac{d}{d\tau}\Big(I_\nu^- e^{\tau/\mu}\Big) = \left(\frac{dI_\nu^-}{d\tau} + \frac{1}{\mu}I_\nu^-\right) e^{\tau/\mu} = \frac{B_\nu(\tau)}{\mu}e^{\tau/\mu}. \tag{23}$$

- The physical picture in Fig. 12 of downgoing beams which start at the "top" and interact with the medium in the slab on their way downward, suggest that we integrate Eq. 23 along the vertical from the "top" ($\tau = 0$) to the "bottom" ($\tau = \tau^*$) of the medium to obtain:

$$\int_0^{\tau^*} d\tau' \frac{d}{d\tau'}\left(I_\nu^- \, e^{\tau'/\mu}\right) = I_\nu^-(\tau^*, \mu, \phi)e^{\tau^*/\mu} - I_\nu^-(0, \mu, \phi) = \int_0^{\tau^*} \frac{d\tau'}{\mu} e^{\tau'/\mu} B_\nu(\tau').$$

- Solving for $I_\nu^-(\tau^*, \mu, \phi)$, we find:

$$I_\nu^-(\tau^*, \mu, \phi) = I_\nu^-(0, \mu, \phi)e^{-\tau^*/\mu} + \int_0^{\tau^*} \frac{d\tau'}{\mu} B_\nu(\tau')\, e^{-(\tau^* - \tau')/\mu} \tag{24}$$

for the radiance emerging from the bottom of the slab.

Formal Solution in a Slab Geometry (2)

- For an interior point, $\tau < \tau^*$, we integrate from 0 to τ. The solution is easily found by replacing τ^* by τ in Eq. 24, *i.e.*

$$I_\nu^-(\tau,\mu,\phi) = I_\nu^-(0,\mu,\phi)e^{-\tau/\mu} + \int_0^\tau \frac{d\tau'}{\mu} B_\nu(\tau')\, e^{-(\tau-\tau')/\mu}. \tag{25}$$

- For the upper-half range radiance the integrating factor for Eq. 21 is $e^{-\tau/\mu}$:

$$\frac{d}{d\tau}\left(I_\nu^+ e^{-\tau/\mu}\right) = \left(\frac{dI_\nu^+}{d\tau} - \frac{1}{\mu} I_\nu^+\right) e^{-\tau/\mu} = -\frac{B_\nu(\tau)}{\mu} e^{-\tau/\mu}.$$

- The physical picture (Fig. 12) involves upgoing beams. Therefore, we integrate from the "bottom" to the "top" of the medium:

$$\begin{aligned}\int_{\tau^*}^0 d\tau' \frac{d}{d\tau'}\left(I_\nu^+ e^{-\tau'/\mu}\right) &= I_\nu^+(0,\mu,\phi) - I_\nu^+(\tau^*,\mu,\phi)e^{-\tau^*/\mu} \\ &= -\int_{\tau^*}^0 \frac{d\tau'}{\mu} e^{-\tau'/\mu} B_\nu(\tau') = \int_0^{\tau^*} \frac{d\tau'}{\mu} e^{-\tau'/\mu} B_\nu(\tau').\end{aligned}$$

Formal Solution in a Slab Geometry (3)

- Solving for $I_\nu^+(0,\mu,\phi)$, we find:

$$I_\nu^+(0,\mu,\phi) = I_\nu^+(\tau^*,\mu,\phi)e^{-\tau^*/\mu} + \int_0^{\tau^*} \frac{d\tau'}{\mu} e^{-\tau'/\mu} B_\nu(\tau').$$

- To find the radiance at an interior point τ, we integrate from τ^* to τ, to obtain:

$$\boxed{I_\nu^+(\tau,\mu,\phi) = I_\nu^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu} + \int_\tau^{\tau^*} \frac{d\tau'}{\mu} e^{-(\tau'-\tau)/\mu} B_\nu(\tau'). \qquad (26)}$$

- Integration **along the beam direction** promotes a good physical understanding of the radiative transfer process. Mathematically, the integration direction is irrelevant: either direction gives the same answer.
- What happens when $\mu \to 0$, that is, when the line-of-sight traverses an infinite distance parallel to the slab? Since $B_\nu(\tau)$ is constant in Eqs. 25 and 26:

$$I_\nu^\pm(\tau,\mu=0,\phi) = B_\nu(\tau). \qquad (27)$$

Formal Solution Including Scattering and Emission (1)

- If we cannot neglect scattering, the source function is written (see Eq. 13):

Source Function due to Thermal Emission and Multiple Scattering

$$S(\tau, \hat{\Omega}) = [1 - \varpi(\tau)]B(\tau) + \frac{\varpi(\tau)}{4\pi} \int_{4\pi} d\omega' p(\tau, \hat{\Omega}', \hat{\Omega}) I(\tau, \hat{\Omega}'). \tag{28}$$

We note that the independent variable is the:

- extinction optical depth = sum of absorption and scattering optical depths.
- The source function is generally a function of the direction $\hat{\Omega}$ of the "emitted" beam, and also a function of the local radiance distribution.

The general radiative transfer equation is:

$$\frac{dI(\tau_{\rm s}, \hat{\Omega})}{d\tau_{\rm s}} = -I(\tau_{\rm s}, \hat{\Omega}) + S(\tau_{\rm s}, \hat{\Omega}). \tag{29}$$

Formal Solution Including Scattering and Emission (2)

- Using the method of the integrating factor, we can write the formal solution of Eq. 29 by replacing B with S in Eq. 16:

Radiance in terms of an Integration over the Source Function

$$I[\tau_{\rm s}(P_2), \hat{\Omega}] = I[\tau_{\rm s}(P_1), \hat{\Omega}]e^{-\tau_{\rm s}(P_1,P_2)} + \int_{\tau_{\rm s}(P_1)}^{\tau_{\rm s}(P_2)} d\tau_{\rm s}' \, S(\tau_{\rm s}', \hat{\Omega})e^{-\tau_{\rm s}'(P,P_2)}. \qquad (30)$$

We stress that:

- This solution is only a **formal** solution, since (in contrast to Eq. 16) the source function is unknown: it depends upon the radiation field, as seen in Eq. 28.
- The importance of this 'solution' is that it emphasizes that, apart from boundary terms, **a knowledge of the source function, $S(\tau_{\rm s}, \hat{\Omega})$, is equivalent to knowledge of the complete solution of the radiative transfer problem**.

Formal Solution Including Scattering and Emission (3)

- For a slab the solutions to Eqs. 28 and 30 are given by Eqs. 25, 26, and 27:

$$I^-(\tau,\mu,\phi) = I^-(0,\mu,\phi)e^{-\tau/\mu} + \int_0^{\tau} \frac{d\tau'}{\mu} S(\tau',\mu,\phi)\, e^{-(\tau-\tau')/\mu} \tag{31}$$

$$I^+(\tau,\mu,\phi) = I^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu} + \int_{\tau}^{\tau^*} \frac{d\tau'}{\mu} S(\tau',\mu,\phi)\, e^{-(\tau'-\tau)/\mu} \tag{32}$$

$$I^{\pm}(\tau,\mu=0,\phi) = S(\tau,\mu=0,\phi). \tag{33}$$

The source function is easily derived from Eq. 28:

$$\begin{aligned} S(\tau,\mu,\phi) \; &= \; (1-\varpi)B(\tau) + \frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu' p(\mu',\phi';\mu,\phi) I^+(\tau,\mu',\phi') \\ &+ \; \frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d\mu' p(-\mu',\phi';\mu,\phi) I^-(\tau,\mu',\phi'). \end{aligned} \tag{34}$$

Radiative Heating Rate (1)

- The differential change of energy over the distance ds along a beam is:

$$\delta(d^4E) = dI_\nu \, dA \, dt \, d\nu \, d\omega.$$

Dividing this expression by $dsdA = dV$, and also by $d\nu dt$, we obtain:

- The time rate of change in radiative energy per unit volume per unit frequency, due to the change in radiance for beams within the solid angle $d\omega$.

- Since there is (generally) incoming radiation from **all** directions, the total change in energy per unit frequency per unit time per unit volume is:

$$\int_{4\pi} d\omega \frac{dI_\nu}{ds} = \int_{4\pi} d\omega \left(\hat{\Omega} \cdot \nabla I_\nu\right).$$

- The **spectral radiative heating rate** $\mathcal{H}_\nu$ is (minus) the rate of change of the radiative energy per unit volume:

$$\mathcal{H}_\nu = -\int_{4\pi} d\omega \left(\hat{\Omega} \cdot \nabla I_\nu\right).$$

Radiative Heating Rate (2)

The **net radiative heating rate**, $\mathcal{H}$ is

$$\mathcal{H} = -\int_0^\infty d\nu \int_{4\pi} d\omega \left(\hat{\Omega} \cdot \nabla I_\nu\right). \tag{35}$$

- In slab geometry the radiative heating rate is written:

$$\mathcal{H}_\nu = -\int_0^\infty d\nu \frac{\partial F_\nu}{\partial z} = -2\pi \int_0^\infty d\nu \int_{-1}^{+1} du\, u \frac{\partial I_\nu}{\partial z} \tag{36}$$

where $F_\nu = F_\nu^+ - F_\nu^-$ is the irradiance in the z direction (Eq. 20).

Now, recall the RTE (Eq. 13):

$$\frac{dI_\nu}{d\tau_s} = -I_\nu + \underbrace{[1 - \varpi(\nu)]B_\nu(T)}_{\text{thermal emission}} + \underbrace{\frac{\varpi(\nu)}{4\pi} \int_{4\pi} d\omega'\, p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}')}_{\text{multiple scattering}}.$$

Generalized Gershun's Law (1)

Substituting $dI_\nu/d\tau_s = -u dI_\nu/d\tau$ in Eq. 36 from the radiative transfer equation [Eq. 13: $\frac{dI_\nu}{d\tau_s} = -I_\nu + [1 - \varpi(\nu)]B_\nu(T) + \frac{\varpi(\nu)}{4\pi}\int_{4\pi} d\omega'\, p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}')$], and using $k(\nu) = \sigma(\nu) + \alpha(\nu)$, we obtain:

$$\mathcal{H}_\nu = 4\pi\alpha(\nu)\bar{I}_\nu + 4\pi\sigma(\nu)\bar{I}_\nu - 4\pi\alpha(\nu)B_\nu(T) - \overbrace{\sigma(\nu)\int_{4\pi} d\omega' \underbrace{\int_{4\pi}\frac{d\omega}{4\pi} p(\hat{\Omega}', \hat{\Omega})}_{=1} I_\nu(\hat{\Omega}')}^{4\pi\sigma(\nu)\bar{I}_\nu}$$

where $\bar{I}_\nu$ is the angular average of the radiance: $\bar{I}_\nu = \int_{4\pi} d\omega' I_\nu(\hat{\Omega}')/4\pi$.

- Employing the normalization condition for the phase function, we see that the two scattering terms cancel. Integration over frequencies therefore yields:

The radiative heating rate is the rate at which radiative energy is absorbed, less the rate at which it is emitted:

$$\mathcal{H} = -\nabla \cdot \vec{F} = 4\pi \int_0^\infty d\nu \alpha(\nu)\bar{I}_\nu - 4\pi \int_0^\infty d\nu \alpha(\nu) B_\nu(T). \qquad (37)$$

- When internal emission is absent, Eq. 37 is known as **Gershun's Law**.

Generalized Gershun's Law (2)

- When $\mathcal{H} = 0$, the volume absorption rate exactly balances the volume emission rate.
- This situation may happen locally at points where the net heating rate happens to change sign, but if the **entire** medium experiences this balance, we have the condition of **planetary radiative equilibrium**.
- Clearly, if there is no absorption anywhere in the medium, then $\mathcal{H} = 0$ everywhere. In a slab medium, radiative equilibrium implies that $\partial F/\partial z = 0$ and thus $F =$ constant.