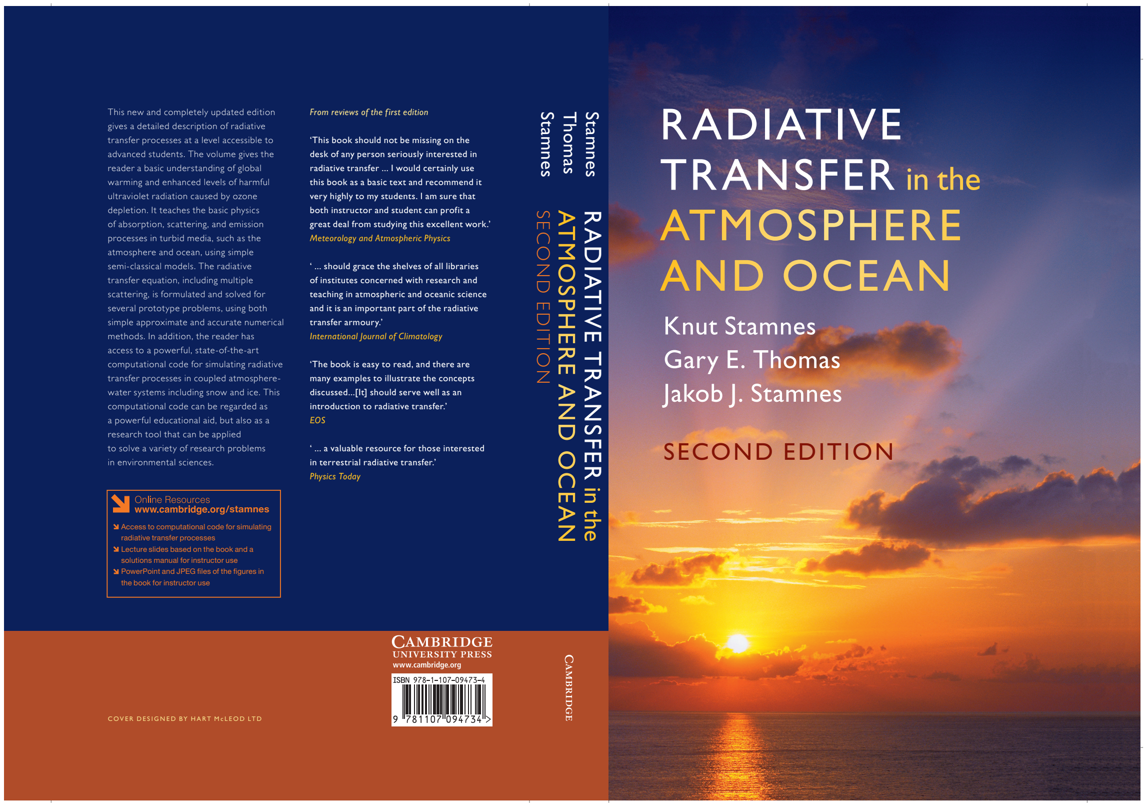
# Lecture *Notes:*
# Formulation of Radiative Transfer Problems – I




**Based on Chapter 6 in K. Stamnes, G. E. Thomas, and J. J. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, 2017.**

# Separation into Diffuse and Direct (Solar) Components (1)

In full-range slab geometry the radiance is found by solving $[I_\nu(\tau, u, \phi) \equiv I_\nu(\tau, \hat{\Omega})]$

$$
\begin{aligned}
u\frac{dI_\nu(\tau,u,\phi)}{d\tau} &= I_\nu(\tau,u,\phi) - \frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I_\nu(\tau,u',\phi') \\
&\quad - (1-\varpi)B_\nu. \qquad (1)
\end{aligned}
$$

Recall that in half-range geometry, we defined

$$
\begin{aligned}
I_\nu^+(\tau,\hat{\Omega}) = I_\nu^+(\tau,\theta,\phi) &\equiv I_\nu(\tau,\theta \le \pi/2,\phi) \\
I_\nu^-(\tau,\hat{\Omega}) - I_\nu^-(\tau,\theta,\phi) &\equiv I_\nu(\tau,\theta > \pi/2,\phi).
\end{aligned}
$$

The shortwave radiation field consists of two components:

- The direct or **solar** component, $I_{\nu \mathrm{s}}$, which is that part of the solar radiation field having survived extinction, i.e.,

$$
I_{\nu\mathrm{s}}^-(\tau,\mu,\phi) = F_\nu^{\mathrm{s}} e^{-\tau/\mu_0}\delta(\hat{\Omega}-\hat{\Omega}_0) = F_\nu^{\mathrm{s}} e^{-\tau/\mu_0}\delta(\mu-\mu_0)\delta(\phi-\phi_0). \qquad (2)
$$

- The **diffuse** component, $I_{\mathrm{d}}$, which consists of light that has been scattered at least once. This part is also called the **multiple-scattering** component.

# Separation into Diffuse and Direct (Solar) Components (2)

Since the direct component is described by the extinction law, we may isolate this part from the total radiation field. We start by writing:

$$I_\nu^-(\tau,\mu,\phi) = I_{\nu\mathrm{d}}^-(\tau,\mu,\phi) + I_{\nu\mathrm{s}}^-(\tau,\mu,\phi) \quad \longleftarrow \quad \textbf{diffuse} + \textbf{solar}. \tag{3}$$

Let the lower surface at be black and ignore thermal emission from the surface, but include thermal radiation from the medium itself. Then:

$$I_{\nu\mathrm{s}}^+(\tau^*,\mu,\phi) = 0, \qquad \text{and} \qquad I_\nu^+(\tau,\mu,\phi) = I_{\nu\mathrm{d}}^+(\tau,\mu,\phi)$$

where $\tau^*$ denotes the total optical depth of the medium.

In slab geometry $[\hat{\Omega} = (\mu,\phi), \hat{\Omega}' = (\mu',\phi')]$ the RTE for $I_\nu^\pm(\tau,\hat{\Omega})$ becomes:

$$\begin{aligned} -\mu \frac{dI_\nu^-(\tau,\hat{\Omega})}{d\tau} = \; & I_\nu^-(\tau,\hat{\Omega}) - (1-\varpi)B_\nu - \frac{\varpi}{4\pi}\int_+ d\omega' p(+\hat{\Omega}',-\hat{\Omega}) I_\nu^+(\tau,\hat{\Omega}') \\ & - \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',-\hat{\Omega}) I_\nu^-(\tau,\hat{\Omega}') \end{aligned} \tag{4}$$

$$\begin{aligned} \mu \frac{dI_\nu^+(\tau,\hat{\Omega})}{d\tau} = \; & I_\nu^+(\tau,\hat{\Omega}) - (1-\varpi)B_\nu - \frac{\varpi}{4\pi}\int_+ d\omega' p(+\hat{\Omega}',+\hat{\Omega}) I_\nu^+(\tau,\hat{\Omega}') \\ & - \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',+\hat{\Omega}) I_\nu^-(\tau,\hat{\Omega}'). \end{aligned} \tag{5}$$

## Separation into Diffuse and Direct (Solar) Components (3)

Here $I_\nu^-(\tau,\mu,\phi) \equiv I_\nu^-(\tau,\hat{\Omega}) \equiv I_\nu(\tau,-\hat{\Omega})$ and

$p(-\hat{\Omega}',+\hat{\Omega}) \longrightarrow$ photon is moving downward $(-\hat{\Omega}')$ before, and upward $(+\hat{\Omega})$ after the scattering.

Substituting for the total radiance field, the sum of the diffuse and the direct or solar components $[I_\nu^-(\tau,\mu,\phi) = I_{\nu\mathrm{d}}^-(\tau,\mu,\phi) + I_{\nu\mathrm{s}}^-(\tau,\mu,\phi)]$, into Eq. 4, we obtain:

$$-\mu\frac{dI_{\nu\mathrm{d}}^-(\tau,\hat{\Omega})}{d\tau} - \mu\frac{dI_{\nu\mathrm{s}}^-(\tau,\hat{\Omega})}{d\tau} =$$

$$I_{\nu\mathrm{d}}^-(\tau,\hat{\Omega}) + I_{\nu\mathrm{s}}^-(\tau,\hat{\Omega}) - (1-\varpi)B_\nu - \overbrace{\frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',-\hat{\Omega}) I_{\nu\mathrm{s}}^-(\tau,\hat{\Omega}')}^{S_\nu^*(\tau,-\hat{\Omega})}$$
$$-\frac{\varpi}{4\pi}\int_+ d\omega' p(+\hat{\Omega}',-\hat{\Omega}) I_{\nu\mathrm{d}}^+(\tau,\hat{\Omega}') - \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',-\hat{\Omega}) I_{\nu\mathrm{d}}^-(\tau,\hat{\Omega}'). \tag{6}$$

**The two non-integral terms involving the direct (solar) component cancel**, because $-\mu dI_{\nu\mathrm{s}}^-/d\tau = I_{\nu\mathrm{s}}^-$.

# Separation into Diffuse and Direct (Solar) Components (4)

If we substitute for $I^-_{\nu\mathrm{s}}$ from Eq. 2 $[I^-_{\nu\mathrm{s}}(\tau,\mu,\phi) = F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}\delta(\hat{\Omega}-\hat{\Omega}_0) = F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}\delta(\mu-\mu_0)\delta(\phi-\phi_0)]$ in the first integral term, we obtain the result:

$$\begin{aligned} -\mu\frac{dI^-_{\nu\mathrm{d}}(\tau,\hat{\Omega})}{d\tau} &= I^-_{\nu\mathrm{d}}(\tau,\hat{\Omega}) - (1-\varpi)B_\nu - S^*_\nu(\tau,-\hat{\Omega}) \\ &- \frac{\varpi}{4\pi}\int_+ d\omega' p(\hat{\Omega}',-\hat{\Omega})I^+_{\nu\mathrm{d}}(\tau,\hat{\Omega}') - \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',-\hat{\Omega})I^-_{\nu\mathrm{d}}(\tau,\hat{\Omega}') \end{aligned} \tag{7}$$

where $[\hat{\Omega} = (\mu,\phi)]$

$$\begin{aligned} S^*_\nu(\tau,-\hat{\Omega}) &= \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',-\hat{\Omega})F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}\delta(\hat{\Omega}'-\hat{\Omega}_0) \\ &= \frac{\varpi}{4\pi}p(-\hat{\Omega}_0,-\hat{\Omega})F^{\mathrm{s}}_\nu e^{-\tau/\mu_0} \\ &= \frac{\varpi}{4\pi}p(-\mu_0,\phi_0;-\mu,\phi)F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}. \end{aligned} \tag{8}$$

# Separation into Diffuse and Direct (Solar) Components (5)

We repeat this procedure for the upward component (Eq. 5) to obtain:

$$\mu \frac{dI^+_{\nu\mathrm{d}}(\tau,\hat{\Omega})}{d\tau} = I^+_{\nu\mathrm{d}}(\tau,\hat{\Omega}) - (1-\varpi)B_\nu - S^*_\nu(\tau,+\hat{\Omega})$$
$$- \frac{\varpi}{4\pi}\int_+ d\omega' p(+\hat{\Omega}',+\hat{\Omega}) I^+_{\nu\mathrm{d}}(\tau,\hat{\Omega}') - \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',+\hat{\Omega}) I^-_{\nu\mathrm{d}}(\tau,\hat{\Omega}') \qquad (9)$$

where $[\hat{\Omega} = (\mu,\phi)]$

$$S^*_\nu(\tau,+\hat{\Omega}) = \frac{\varpi}{4\pi}\int_- d\omega' p(-\hat{\Omega}',+\hat{\Omega}) F^{\mathrm{s}}_\nu e^{-\tau/\mu_0} \delta(\hat{\Omega}'-\hat{\Omega}_0)$$
$$= \frac{\varpi}{4\pi} p(-\hat{\Omega}_0,+\hat{\Omega}) F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}$$
$$= \frac{\varpi}{4\pi} p(-\mu_0,\phi_0;+\mu,\phi) F^{\mathrm{s}}_\nu e^{-\tau/\mu_0}. \qquad (10)$$

# Separation into Diffuse and Direct (Solar) Components (6)

The equations of transfer for the total field, and for the diffuse field differ by the presence of an extra, single-scattering "source" term $S_\nu^*(\tau, \pm\hat{\Omega})$:

- This **single-scattering source term "drives"** the diffuse radiation field.
- Without $S_\nu^*(\tau, \pm\hat{\Omega})$ there would be no diffuse radiation (if $B_\nu = 0$).
- Note also that **the azimuthal dependence of the radiation field can be traced to that of** $S_\nu^*(\tau, \pm\hat{\Omega})$ through the phase function $p(-\hat{\Omega}_0, \pm\hat{\Omega})$.

In full-range slab geometry the radiative transfer equation for the diffuse radiance may be written more compactly as:

$$\begin{aligned} u\frac{dI_{\nu\mathrm{d}}(\tau, u, \phi)}{d\tau} \;=\; & I_{\nu\mathrm{d}}(\tau, u, \phi) - \frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u', \phi'; u, \phi) I_\nu(\tau, u', \phi') \\ & - (1-\varpi)B_\nu - S_\nu^*(\tau, u, \phi) \end{aligned} \tag{11}$$

where $S_\nu^*(\tau, u, \phi)$ denotes the solar beam driving term derived above:

$$S_\nu^*(\tau, u, \phi) = \frac{\varpi}{4\pi} p(-\mu_0, \phi_0; u, \phi) F_\nu^{\mathrm{s}} e^{-\tau/\mu_0}.$$

# Single versus Multiple Scattering

The interpretation of Eqs. 7–10 for the diffuse radiance is straightforward:

- The extra term $S^*$ (dropping the $\nu$ subscript) is an **"imbedded source"** of radiation which has been **scattered once** (first-order or single scattering) within the medium.

- The integral terms constitute the source of **multiply-scattered** radiation.

Thus, the total source function consists of the following sum:

$$S(\tau,\hat{\Omega}) = \underbrace{[1-\varpi]B(T)}_{\text{thermal emission}} + \underbrace{S^*(\tau,\hat{\Omega})}_{\text{first-order scattering}} + \underbrace{\frac{\varpi}{4\pi}\int_{4\pi} d\omega' p(\hat{\Omega}',\hat{\Omega}) I_{\rm d}(\hat{\Omega}')}_{\text{multiple scattering}}. \tag{12}$$

Here $S_\nu(\tau,\hat{\Omega})$ refers to the sources of all *internal* radiation.

# Example: Isotropic Scattering in Slab Geometry

Assume that the scattering is isotropic, so that $p(\hat{\Omega}', \hat{\Omega}) = 1$. The source term is then **isotropic**: $S_\nu^{*\pm}(\tau, \hat{\Omega}) = \frac{\varpi}{4\pi} p(-\hat{\Omega}_0, \pm\hat{\Omega}) F_\nu^{\rm s} e^{-\tau/\mu_0} = \frac{\varpi}{4\pi} F_\nu^{\rm s} e^{-\tau/\mu_0} = S^*(\tau)$, and:

- **The RT equations for the half-range diffuse radiance fields are greatly simplified because the integrals are independent of $\phi$.**

Dropping subscripts '$\nu$' and 'd', and assuming a black lower boundary, we find*

$$\mu \frac{dI^+(\tau,\mu)}{d\tau} = I^+(\tau,\mu) - (1-\varpi)B - S^*(\tau) - \frac{\varpi}{2} \int_0^1 d\mu' I^+(\tau,\mu') - \frac{\varpi}{2} \int_0^1 d\mu' I^-(\tau,\mu') \tag{13}$$

$$-\mu \frac{dI^-(\tau,\mu)}{d\tau} = I^-(\tau,\mu) - (1-\varpi)B - S^*(\tau) - \frac{\varpi}{2} \int_0^1 d\mu' I^+(\tau,\mu') - \frac{\varpi}{2} \int_0^1 d\mu' I^-(\tau,\mu') \tag{14}$$

where $S^*(\tau) = \frac{\varpi}{4\pi} F^s e^{-\tau/\mu_0}$ $\longleftarrow$ **isotropic**.

Because $S^*(\tau)$ is isotropic in this case:

- **The radiances are also independent of the angle $\phi$, which is an enormous simplification over the anisotropic scattering case.**



*The corresponding equation for the full range is: $u \frac{dI(\tau,u)}{d\tau} = I(\tau,u) - \frac{\varpi}{4\pi} \int_{-1}^1 du' p(u',u) I(\tau,u') - S^*(\tau)$.

# Azimuth-Independence of Irradiance and Mean Radiance (1)

We now prove an important result:

- **In slab geometry, the irradiances and the mean radiance depend only on the azimuthally averaged radiance.**

By averaging Eqs. 7 and 9 over azimuth, we obtain the following pair of equations for the azimuthally averaged half-range diffuse radiances (applying operator $(1/2\pi)\int_0^{2\pi} d\phi \cdots$):

$$\mu \frac{dI^+(\tau,\mu)}{d\tau} = I^+(\tau,\mu) - (1-\varpi)B - \frac{\varpi}{2}\int_0^1 d\mu' p(+\mu',+\mu) I^+(\tau,\mu') - \frac{\varpi}{2}\int_0^1 d\mu' p(-\mu',+\mu) I^-(\tau,\mu') - S^*(\tau,\mu) \quad (15)$$

$$-\mu \frac{dI^-(\tau,\mu)}{d\tau} = I^-(\tau,\mu) - (1-\varpi)B - \frac{\varpi}{2}\int_0^1 d\mu' p(+\mu',-\mu) I^+(\tau,\mu') - \frac{\varpi}{2}\int_0^1 d\mu' p(-\mu',-\mu) I^-(\tau,\mu') - S^*(\tau,-\mu) \quad (16)$$

# Azimuth-Independence of Irradiance and Mean radiance (2)

where we have introduced the azimuthally averaged radiance:

$$I^{\pm}(\tau,\mu) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi' I^{\pm}(\tau,\mu,\phi') \tag{17}$$

the azimuthally averaged scattering phase function:

$$p(\pm\mu',\pm\mu) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi' p(\pm\mu',\phi';\pm\mu,\phi') \equiv \frac{1}{2\pi}\int_0^{2\pi} d(\phi'-\phi) p(\pm\mu',\pm\mu;\phi'-\phi) \tag{18}$$

and the azimuthally averaged source function:

$$S^{*}(\tau,\pm\mu) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' S^{*}(\tau,\pm\mu,\phi') = \frac{\varpi}{4\pi} p(-\mu_0,\pm\mu) F^{\mathrm{s}} e^{-\tau/\mu_0}. \tag{19}$$

The presence of $S^*$ is always the clue that we are referring to the diffuse radiance. By definition:

$$\begin{aligned} F^{\pm}(\tau) &\equiv \int_0^{2\pi} d\phi' \int_0^1 d\mu'\ \mu'\ I^{\pm}(\tau,\mu',\phi') = \\ &2\pi \int_0^1 d\mu'\ \mu' \frac{1}{2\pi}\int_0^{2\pi} d\phi'\ I^{\pm}(\tau,\mu',\phi') = 2\pi\int_0^1 d\mu'\ \mu'\ I^{\pm}(\tau,\mu'). \end{aligned} \tag{20}$$

# Azimuth-Independence of Irradiance and Mean Radiance (3)

We have used the absence of $\phi$-arguments to indicate independence of azimuth angle.

Hence, we see that:

- **In slab geometry the irradiance, $F^{\pm}(\tau)$, depends only on the azimuthally averaged radiance, $I^{\pm}(\tau, \mu)$.**

Thus, if we are interested only in irradiance (as opposed to angular-dependent radiances):

- We need to consider only the **azimuthally independent component** of the radiance.
- Similarly, we find that **the mean radiance depends only on the azimuthally averaged radiance.**

# Azimuth-Independence of Irradiance and Mean Radiance (4)

Switching back to the $u$-coordinate, we have:

$$\begin{aligned}\bar{I}(\tau) &= \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' I(\tau, u', \phi') = \frac{1}{2}\int_{-1}^{1} du' \frac{1}{2\pi}\int_0^{2\pi} d\phi' I(\tau, u', \phi') \\ &= \frac{1}{2}\int_{-1}^{1} du' I(\tau, u'). \end{aligned} \tag{21}$$

Finally, we may integrate Eqs. 4 and 5 (or Eq. 1 with $S^*(\tau, u, \phi) = 0$) over $4\pi$ steradians so that we consider the total (diffuse plus direct) radiation field. The result is:

$$\begin{aligned}\frac{dF}{d\tau} &= 4\pi(1-\varpi)(\bar{I} - B) \quad \longleftarrow \quad \textbf{net irradiance divergence} \\ &= (1-\varpi)F^{\mathrm{s}}e^{-\tau/\mu_0} + 4\pi(1-\varpi)(\bar{I}_{\mathrm{d}} - B)\end{aligned} \tag{22}$$

which shows that **a constant net irradiance ($F$ = constant)** is obtained if:

- There is no absorption in the medium ($\varpi = 1$)
- The slab is in **monochromatic radiative equilibrium** ($\bar{I} = B$).

Eq. (22) is proportional to the **spectral heating rate**.

# Azimuthal Dependence of the Radiation Field (1)

If only irradiances or heating rates are desired:

- a radiative transfer problem involving only **two** variables, $\tau$ and $u$ must be solved

However, if we desire the radiance or the source function:

- we have to deal with a function of **three** variables, $\tau$, $u$, and $\phi$.

We will show that:

- by using an **Addition Theorem** we can reduce the problem to solving for only **two** variables, also when we desire radiances.

**Below we describe a transformation that reduces the problem to one of solving a finite set of uncoupled radiative transfer equations, each of which depends on only two variables, $\tau$ and $u$.**

We start by expanding the scattering phase function in a finite series of $2N$ **Legendre polynomials** as follows:

# Azimuthal Dependence of the Radiation Field (2)

$$p(\tau, \hat{\Omega}', \hat{\Omega}) = p(\tau, \hat{\Omega}' \cdot \hat{\Omega}) = p(\tau, \cos\Theta) \approx \sum_{\ell=0}^{2N-1} (2\ell+1)\chi_l(\tau) P_\ell(\cos\Theta) \tag{23}$$

where $P_\ell$ is the $\ell^{th}$ *Legendre polynomial*. The $\ell^{th}$ expansion coefficient is given by:

$$\chi_\ell(\tau) = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta) P_\ell(\cos\Theta) p(\tau, \cos\Theta). \tag{24}$$

It is common to denote the first moment of the phase function by the symbol $g \equiv \chi_1$. (Another notation is $\langle\cos\Theta\rangle$.)

The first moment represents the degree of asymmetry of the angular scattering and is therefore called the **asymmetry factor**. Special values for the asymmetry factor are given below:

$$\begin{aligned}
g &= 0 & \longleftarrow & \quad \text{isotropic scattering, or symmetric about } \cos\Theta = 0 \\
g &= -1 & \longleftarrow & \quad \text{complete backscattering} \\
g &= 1 & \longleftarrow & \quad \text{complete forward scattering.}
\end{aligned}$$

The probability of scattering into the backward hemisphere is given by the *backscattering ratio* ($y = -\cos\Theta$, $dy = \sin\Theta d\Theta$):

$$b = \frac{1}{2}\int_{\pi/2}^{\pi} d\Theta \sin\Theta p(\tau, \cos\Theta) = \frac{1}{2}\int_{0}^{1} dy p(\tau, -y). \tag{25}$$

# Azimuthal Dependence of the Radiation Field (3)

The *Legendre polynomials* $P_\ell(u = \cos\Theta)$ are a natural basis set of orthogonal polynomials over the angular domain ($0^\circ \leq \Theta \leq 180^\circ$ or ; $-1 \leq u \leq 1$). The first few *Legendre polynomials* are

$$\begin{aligned} P_0(u) &= 1; \qquad P_1(u) = u; \qquad P_2(u) = \frac{1}{2}(3u^2 - 1) \\ P_3(u) &= \frac{1}{2}(5u^3 - 3u); \qquad P_4(u) = \frac{1}{8}(35u^4 - 30u^2 + 3). \end{aligned}$$

An important property of the *Legendre polynomials* is that they are *orthogonal* to one another

$$\frac{1}{2}\int_{-1}^{+1} du P_\ell(u) P_k(u) = \frac{1}{2\ell + 1}\delta_{\ell k}. \tag{26}$$

Here, $\delta_{\ell k}$ is the *Kronecker delta* ($\delta_{\ell k} = 1$ for $\ell = k$ and $\delta_{\ell k} = 0$ for $\ell \neq k$).

**The number of terms, $2N$, in the expansion required for an accurate representation of $p(\tau, \cos\Theta)$ depends on how asymmetric the phase function is.**

A one-parameter phase function first proposed by the astronomers Henyey and Greenstein in 1941 is

$$p_{\mathrm{HG}}(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}}.$$

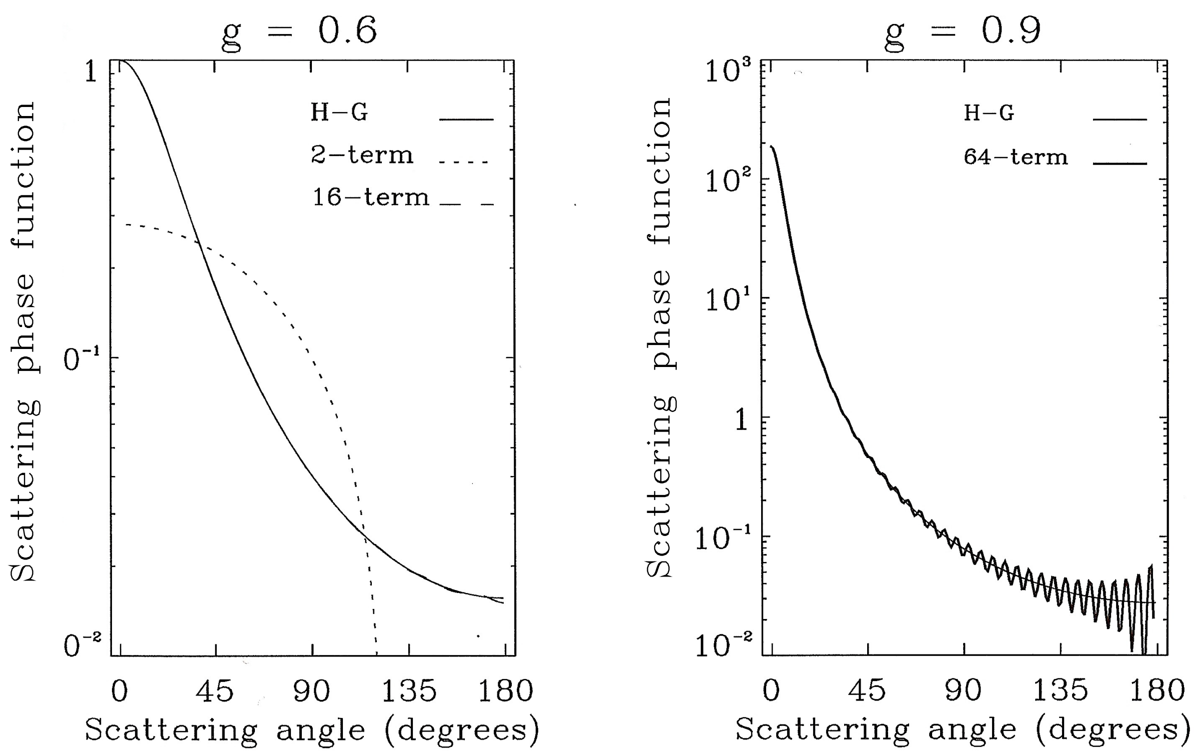



Figure 1: Illustration of Legendre polynomial fit to the synthetic Henyey-Greenstein (HG) scattering phase function for two values of the asymmetry factor $g = \chi_1$ as indicated. The larger the value of $g$ the more anisotropic the phase function. Less anisotropic scattering phase functions require fewer Legendre polynomial expansion terms to obtain a reasonable fit.

# Azimuthal Dependence of the Radiation Field (4)

- For isotropic scattering ($p(\cos\Theta) \approx \Sigma_{\ell=0}^{2N-1}(2\ell+1)\chi_l(\tau)P_\ell(\cos\Theta) = 1$) only *one* term is needed: $\chi_0 = 1$ and $\chi_\ell = 0$ for $\ell = 1, 2, 3 \ldots, 2N$.
- In general, the more asymmetric the phase function the more terms are required for an accurate representation.
- The $\Theta$-representation is of little use in the transfer equation, which employs polar coordinates $\theta$ and $\phi$ measured with respect to the vertical axis.

The relationship between the polar and azimuthal angles ($\theta', \phi'$ before scattering and $\theta, \phi$ after scattering) and the scattering angle, $\Theta$, is given by ($u = \cos\theta$):

$$\cos\Theta = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi' - \phi) = u'u + \sqrt{1 - u'^2}\sqrt{1 - u^2}\cos(\phi' - \phi).$$

However, use of this relationship in the **Legendre polynomials** yields a rather complicated and useless form. The key in simplifying the expansion of the scattering phase function is the **Addition Theorem for Spherical Harmonics**:

$$\begin{aligned} P_\ell(\cos\Theta) &= P_\ell(u')P_\ell(u) + 2\sum_{m=1}^{\ell}\Lambda_\ell^m(u')\Lambda_\ell^m(u)\cos m(\phi' - \phi) \\ &= \sum_{m=0}^{\ell}(2 - \delta_{0m})\Lambda_\ell^m(u')\Lambda_\ell^m(u)\cos m(\phi' - \phi). \end{aligned} \tag{27}$$

# Azimuthal Dependence of the Radiation Field (5)

To simplify the formulas we have introduced the **normalized associated Legendre polynomial** defined by

$$\Lambda_\ell^m(u) \equiv \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(u) \tag{28}$$

where $P_\ell^m(u)$ is the **associated Legendre polynomial**. The following orthogonality properties apply

$$\frac{1}{2}\int_{-1}^{1} du P_\ell^m(u) P_k^m(u) = \frac{(\ell+m)!}{(2\ell+1)(\ell-m)!}\delta_{\ell k}$$

or

$$\frac{1}{2}\int_{-1}^{1} du \Lambda_\ell^m(u) \Lambda_k^m(u) = \frac{\delta_{\ell k}}{2\ell+1}.$$

If $m = 0$, $\Lambda_\ell^0(u) = P_\ell^0(u) \equiv P_\ell(u)$, the *Legendre polynomial*. Note that we have defined $\Lambda_\ell^m(u)$ in such a way that it satisfies the same orthogonality condition as $P_\ell(u)$.

- Thus, the function $\sqrt{(2\ell+1)/2}\ \Lambda^m(u)$ is orthonormal with respect to the polar angle $\theta$.

# Azimuthal Dependence of the Radiation Field (6)

The first few associated **Legendre Polynomials** are:

$$P_1^1(u) = \sqrt{1-u^2}; \qquad P_1^2(u) = 3u\sqrt{1-u^2}; \qquad P_2^2(u) = 3\sqrt{1-u^2}$$
$$P_3^2(u) = 15u(1-u^2); \qquad P_3^1(u) = \frac{3}{2}\sqrt{1-u^2}(5u^2-1).$$

Application of azimuthal averaging, i.e., $\frac{1}{2\pi}\int_0^{2\pi} d\phi \cdots$, to both sides of (23) gives

$$p(\tau,u',u) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, p(\tau,\cos\Theta) \approx \sum_{l=0}^{2N-1}(2\ell+1)\chi_\ell(\tau)P_\ell(u)P_\ell(u'), \tag{29}$$

where we have made use of Eq. 27 (see also Eq. 33). From Eq. 29 it follows that

$$\frac{1}{2}\int_{-1}^{1} p(\tau,u',u)P_k(u')du' \approx \sum_{\ell=0}^{2N-1}(2\ell+1)\chi_\ell(\tau)P_\ell(u)\frac{1}{2}\int_{-1}^{1} P_\ell(u')P_k(u')du', \tag{30}$$

which by the use of orthogonality (Eq. 26) leads to

$$\chi_\ell(\tau) = \frac{1}{P_\ell(u)}\,\frac{1}{2}\int_{-1}^{1} p(\tau,u',u)P_\ell(u')du'. \tag{31}$$

Thus, to calculate the moments (expansion coefficients), we can use the azimuthally averaged scattering phase function, $p(\tau,u',u)$.

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# Azimuthal Dependence of the Radiation Field (7)

The addition theorem allows us to express the scattering phase function as:

**Legendre polynomial expansion of scattering phase function:**

$$\begin{aligned} p(\cos\Theta) &= p(u', \phi'; u, \phi) \\ &= \sum_{\ell=0}^{2N-1} (2\ell+1)\chi_\ell \sum_{m=0}^{\ell} (2-\delta_{0m}) \Lambda_\ell^m(u') \Lambda_\ell^m(u) \cos m(\phi' - \phi). \end{aligned} \quad (32)$$

Inverting the order of the summation, we have:

$$p(u', \phi'; u, \phi) = \sum_{m=0}^{2N-1} (2-\delta_{0m}) p^m(u', u) \cos m(\phi' - \phi) \quad (33)$$

$$p^m(u', u) = \sum_{\ell=m}^{2N-1} (2\ell+1)\chi_l \Lambda_\ell^m(u') \Lambda_\ell^m(u). \quad (34)$$

Since the expansion of the scattering phase function in Eq. 33 is essentially a **Fourier cosine series**, we should expand the radiance in a similar way:

$$I(\tau, u, \phi) = \sum_{m=0}^{2N-1} I^m(\tau, u) \cos m(\phi_0 - \phi). \quad (35)$$

# Azimuthal Dependence of the Radiation Field (8)

Recall: in full-range slab geometry the radiative transfer equation for the diffuse radiance may be written as:

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') - (1-\varpi)B - S^*(\tau,u,\phi). \tag{36}$$

If we substitute Eqs. 33 and 35 into Eq. 36, we obtain the following equation for each of the Fourier components (using $\Lambda_\ell^m(-u) = (-1)^{\ell+m}\Lambda_\ell^m(u)$):

$$\begin{aligned} u\frac{dI^m(\tau,u)}{d\tau} &= I^m(\tau,u) - (1-\varpi)B\delta_{0m}, \quad m = 0,1,\ldots,2N-1 \\ &- \frac{\varpi}{2}\int_{-1}^{1} du' p^m(\tau,u',u) I^m(\tau,u') - X_0^m(\tau,u)e^{-\tau/\mu_0} \end{aligned} \tag{37}$$

$$\begin{aligned} X_0^m(\tau,u) &= \frac{\varpi}{4\pi}F^{\mathrm{s}}(2-\delta_{0m})p^m(\tau,-\mu_0,u) \\ &= \frac{\varpi}{4\pi}F^{\mathrm{s}}(2-\delta_{0m})\sum_{\ell=m}^{2N-1}(-1)^{\ell+m}(2\ell+1)\chi_\ell\Lambda_\ell^m(u)\Lambda_\ell^m(\mu_0). \end{aligned} \tag{38}$$

# Azimuthal Dependence of the Radiation Field (9)

We have effectively:

- "Isolated" the azimuthal dependence from the radiative transfer equation in the sense that:
- **The various Fourier components in Eq. 37 are entirely uncoupled.**

Thus, in slab geometry:

- **Independent solutions for each $m$ give the azimuthal components, $I^m(\tau, u)$, and the sum:**

$$I(\tau, u, \phi) = \sum_{m=0}^{2N-1} I^m(\tau, u) \cos m(\phi_0 - \phi) \qquad \textbf{(Eq. 35)}$$

**yields the complete azimuthal dependence of the radiance.**

Note that in slab geometry:

- **The azimuthal dependence is forced upon us by the beam source and the boundary conditions.**

# Azimuthal Dependence of the Radiation Field (10)

- **When there is no beam source and no azimuth-dependent reflection at either of the slab boundaries, the sum in Eq. 35** $[I(\tau,u,\phi) = \Sigma_{m=0}^{2N-1} I^m(\tau,u)\cos m(\phi_0-\phi)]$ **reduces to the** $m=0$ **term.**
- Then the angles $\mu_0$ and $\phi_0$ are irrelevant and there is no azimuthal dependence.

We also note that:

- **if the particles scatter isotropically, there is no azimuthal dependence, as we already showed in Eqs. 13–14.**

This behavior follows from Eq. 38 $[X_0^m(\tau,u) = \frac{\varpi}{4\pi}F^{\rm s}(2-\delta_{0m})p^m(\tau,-\mu_0,u)]$, since $X_0^m = 0$ for $m > 0$ if the phase function is set to unity. Finally, we note that:

- **Eq. 37 is of the same mathematical form for all azimuth components** $m$**.** This fact implies that:
- **Any method available for solving the** $m=0$ **azimuth-independent equation can be readily applied to solve the equation for all** $m>0$**.**

# Radiative Transfer in an Atmosphere-Water System (1)

- The **basic radiance** $I/m_{\mathrm{r}}^2$ is invariant along a beam path in the absence of scattering and absorption.
- We have so far assumed that we are dealing with media for which the refractive index is constant throughout the medium. BUT:
- In a coupled atmosphere-water system we must consider **the change in the refractive index across the interface between the atmosphere (with $m_{\mathrm{r}} \approx 1$) and the water (with $m_{\mathrm{r}} \approx 1.34$). HOW do we:**

- **describe radiative transfer throughout a system consisting of two adjacent strata with different refractive indices?**

First: radiative transfer in aquatic media is similar to that in gaseous media:

- In pure aquatic media density fluctuations lead to Rayleigh-like scattering.
- **Turbidity** in an aquatic medium is caused by particles acting to scatter and absorb radiation like aerosol and cloud "particles" do in the atmosphere.

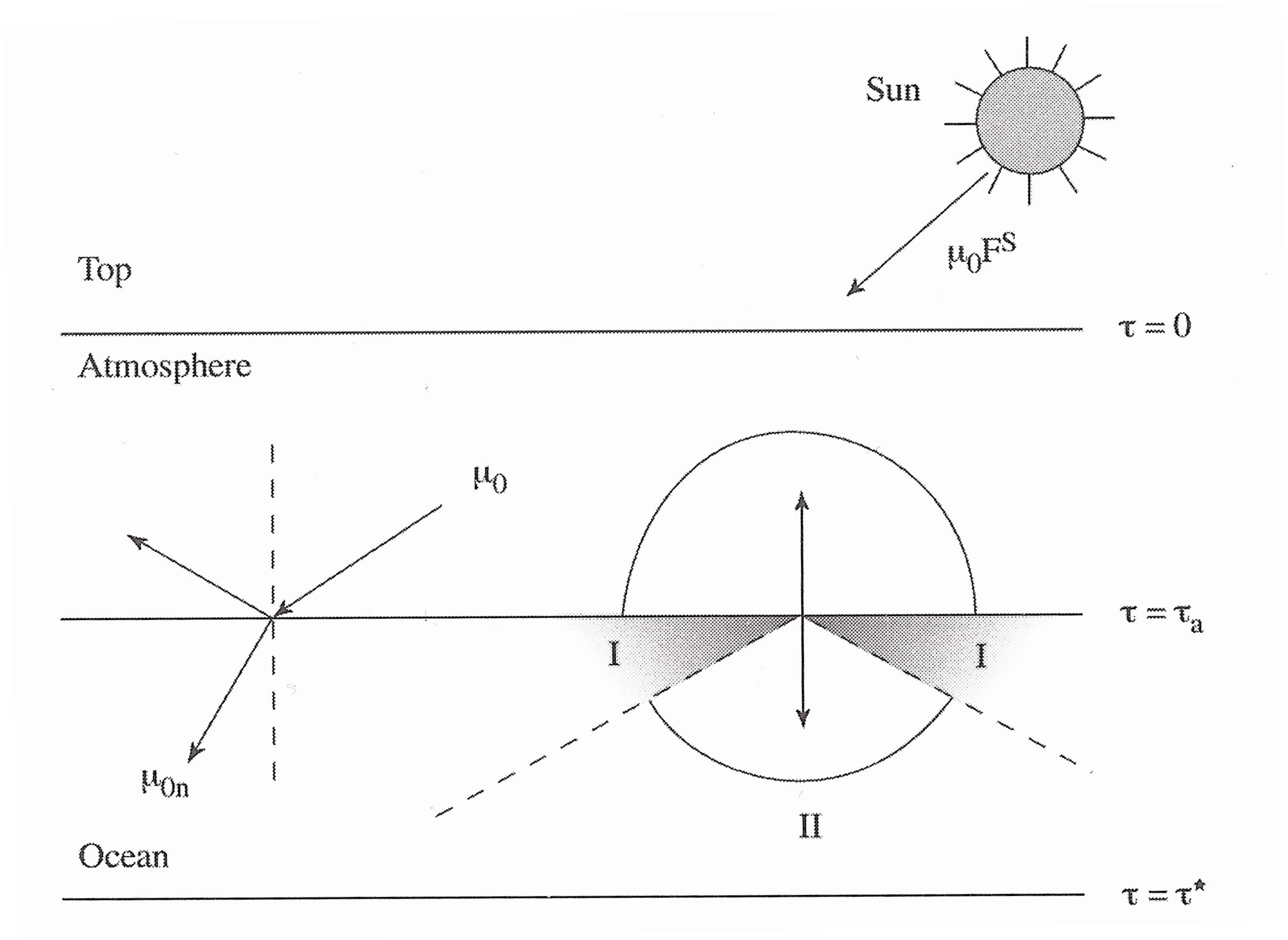


Figure 2: Schematic illustration of two adjacent media with a flat interface such as the atmosphere overlying a calm water body. The atmosphere has a different refractive index ($m_r \approx 1$) than the water ($m_r = 1.34$). Therefore, radiation in the atmosphere distributed over $2\pi$ sr will be confined to a cone less than $2\pi$ sr in the water (region II). Radiation in the water within region I will be totally reflected when striking the interface from below.

# Radiative Transfer in an Atmosphere-Water System (2)

- In the following, we focus on the transfer of solar radiation in an atmosphere-water system as illustrated in Fig. 2. To simplify the situation, we assume that the water surface is calm (i.e., perfectly flat).
- In principle, the radiative coupling of the two media is very simple because:
- It is described by the well-known laws of reflection and refraction that apply at the interface as expressed mathematically by **Snell's Law** and **Fresnel's Equations**.
- The practical complications that arise are due to multiple scattering and total internal reflection.
- The downward radiation distributed over $2\pi$ sr in the atmosphere will be restricted to an angular cone less than $2\pi$ sr (region II in Fig. 2) after being refracted across the interface into the water.
- Beams outside the refractive region in the water are in the total reflection region (referred to as region I hereafter, see Fig. 2).

# Radiative Transfer in an Atmosphere-Water System (3)

- The demarcation between the refractive and total reflective region in the water is given by the critical angle (see Eq. 44), schematically illustrated by the dashed line separating regions I and II in Fig. 2.
- Upward travelling beams in region I in the water will be reflected back into the water upon reaching the interface.
- Thus, beams in region I cannot reach the atmosphere directly (and vice versa); they must be scattered into region II in order to be returned to the atmosphere.

# Two Stratified Media with Different Indices of Refraction (1)

- Since the radiation field in the water is driven by solar radiation passing through the atmosphere:
- we may use the same radiative transfer equation in the water as in the atmosphere as long as we properly incorporate the changes occurring at the atmosphere-water interface.
- Thus, the appropriate radiative transfer equation in either medium is Eq. 1:

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \frac{\varpi(\tau)}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(\tau,u',\phi';u,\phi) I(\tau,u',\phi') - S^*(\tau,u,\phi)$$

where

- $S^*(\tau,u,\phi)$ represents the solar "driving" term.
- This driving term is different in the atmosphere and the water.

# Two Stratified Media with Different Indices of Refraction (2)

- In the atmosphere, we have:

$$S^*_{\rm air}(\tau,u,\phi) = \frac{\varpi(\tau)F^{\rm s}}{4\pi}p(\tau,-\mu_0,\phi_0;u,\phi)e^{-\tau/\mu_0} + \frac{\varpi(\tau)F^{\rm s}}{4\pi}\rho_{\rm s}(-\mu_0;m_{\rm rel})p(\tau,\mu_0,\phi_0;u,\phi)e^{-(2\tau_{\rm a}-\tau)/\mu_0}$$

- $\mu_0$, $\phi_0$, and $F^{\rm s}$ refer to the incident solar beam at the top of the atmosphere;
- $m_{\rm rel} \equiv m_{\rm ocn}/m_{\rm atm}$ is the real index of refraction in the water (ocean) relative to air;
- $\tau_{\rm a}$ is the optical depth of the atmosphere;
- the first term is due to the **usual solar beam source**, whereas the second term is due to **specular reflection by the atmosphere-water interface**.

- To simplify the notation we have written $\rho_{\rm s}(-\mu_0, m_{\rm rel}) \equiv \rho_{\rm s}(-\mu_0,\phi_0;\mu_0,\phi_0+\pi;m_{\rm rel})$ for the specular reflection by the atmosphere-water interface.

# Two Stratified Media with Different Indices of Refraction (3)

- The source term in the water consists of the attenuated solar beam refracted through the interface:

$$S^*_{\rm ocn}(\tau, u, \phi) = \frac{\varpi(\tau) F^{\rm s}}{4\pi} \frac{\mu_0}{\mu_{0m}} \mathcal{T}_{\rm b}(-\mu_0; m_{\rm rel}) p(\tau, -\mu_{0m}, \phi_0; u, \phi) e^{-\tau_{\rm a}/\mu_0} e^{-(\tau-\tau_{\rm a})/\mu_{0m}}$$

where

- $\mathcal{T}_{\rm b}(-\mu_0; m_{\rm rel}) \equiv \mathcal{T}_{\rm b}(-\mu_0, \phi_0; -\mu_{0m}, \phi_0; m_{\rm rel})$ is the beam transmittance through the interface, and

- $\mu_{0m}$ is the cosine of the solar zenith angle in the water, related to $\mu_0$ by **Snell's Law**:

$$\mu_{0m} \equiv \mu_{0m}(\mu_0, m_{\rm rel}) = \sqrt{1 - (1 - \mu_0^2)/m_{\rm rel}^2}.$$

# Two Stratified Media with Different Indices of Refraction (4)

- The "isolation" of the azimuth dependence is now accomplished as usual. The source terms become:

$$S^*_{\rm air}(\tau,u)=X_0^m(\tau,u)e^{-\tau/\mu_0}+X_{01}^m(\tau,u)e^{\tau/\mu_0};\qquad S^*_{\rm ocn}(\tau,u)=X_{02}(\tau,u)e^{-\tau/\mu_{0m}}$$

where

- $X_0^m(\tau,u)$ is given by Eq. 38, and

$$\begin{aligned}X_{01}^m(\tau,u) &= \frac{\varpi(\tau)F^{\rm s}}{4\pi}\rho_{\rm s}(-\mu_0;m_{\rm rel})e^{-2\tau_{\rm a}/\mu_0}(2-\delta_{0m})\\ &\times\sum_{\ell=0}^{2N-1}(2\ell+1)\chi_\ell(\tau)\Lambda_\ell^m(u)\Lambda_\ell^m(\mu_0)\end{aligned}\tag{39}$$

$$\begin{aligned}X_{02}^m(\tau,u) &= \frac{\varpi(\tau)F^{\rm s}}{4\pi}\frac{\mu_0}{\mu_{0m}}\mathcal{T}_{\rm b}(-\mu_0,-\mu_{0m};m_{\rm rel})e^{-\tau_{\rm a}(1/\mu_0-1/\mu_{0m})}(2-\delta_{0m})\\ &\times\sum_{\ell=0}^{2N-1}(-1)^{\ell+m}(2\ell+1)\chi_\ell(\tau)\Lambda_\ell^m(u)\Lambda_\ell^m(\mu_{0m}).\end{aligned}\tag{40}$$

## Two Stratified Media with Different Indices of Refraction (5)

- We must properly account for the reflection from and transmission through the interface. Here the following conditions apply:

$$I_{\rm a}^{+}(\tau_{\rm a},\mu^{\rm a}) = \rho_{\rm s}(-\mu^{\rm a};m_{\rm rel})I_{\rm a}^{-}(\tau_{\rm a},\mu^{\rm a}) + \mathcal{T}_{\rm b}(\mu^{\rm o};m_{\rm rel})[I_{\rm o}^{+}(\tau_{\rm a},\mu^{\rm o})/m_{\rm rel}^{2}] \qquad (41)$$

$$\frac{I_{\rm o}^{-}(\tau_{\rm a},\mu^{\rm o})}{m_{\rm rel}^{2}} = \rho_{\rm s}(\mu^{\rm o};m_{\rm rel})\frac{I_{\rm o}^{+}(\tau_{\rm a},\mu^{\rm o})}{m_{\rm rel}^{2}} + \mathcal{T}_{\rm b}(-\mu^{\rm a};m_{\rm rel})I_{\rm a}^{-}(\tau_{\rm a},\mu^{\rm a}) \quad (\mu^{o} > \mu_{c}) \qquad (42)$$

$$I_{\rm o}^{-}(\tau_{\rm a},\mu^{\rm o}) = I_{\rm o}^{+}(\tau_{\rm a},\mu^{\rm o}) \quad (\mu^{\rm o} < \mu_{\rm c}). \qquad (43)$$

Here:

- $I_{\rm a}(\tau_{\rm a},\mu^{\rm a})$ refers to the radiance in the **atmosphere** evaluated at the interface;
- $I_{\rm o}(\tau_{\rm a},\mu^{\rm o})$ refers to the radiance in the **water** (ocean) evaluated at the interface.

- Equation 41 states that the upward radiance at the interface in the atmosphere consists of the specularly reflected downward atmospheric radiation plus the transmitted upward aquatic radiation.
- Equation 42 states that the downward radiance at the interface in the water consists of the reflected component of aquatic origin plus a transmitted component originating in the atmosphere.
- Equation 43 ensures that radiation in the total reflection region is properly taken into account.

# Two Stratified Media with Different Indices of Refraction (6)

- The demarcation between the refractive and the total reflective region in the water is given by the critical angle, whose cosine is:

$$\mu_c = \sqrt{1 - 1/m_{\rm rel}^2}. \tag{44}$$

- $\mu^o$ and $\mu^a$ are connected through the relation:

$$\mu^o = \mu^o(\mu^a) = \sqrt{1 - [1 - (\mu^a)^2]/m_{\rm rel}^2}.$$

- Note that we have defined $\rho_s(\mu; m_{\rm rel})$ and $\mathcal{T}_b(\mu; m_{\rm rel})$ as the specular reflectance and transmittance of the invariant radiance, $I/m_r^2$, where $m_r$ is the local value of the real part of the refractive index.

- The reflectance and transmittance are derived from Fresnel's equations (see Appendix D).

# Prototype Radiative Transfer Problems (1)

We now describe a few "prototype" radiative transfer problems, which will allow us to compare approximate solutions to "exact" solutions. Also,

- methods deemed to be successful when applied to these prototype problems can be applied with more confidence to more realistic problems.

For each prototype problem, we consider:

- A slab consisting of an optically uniform (homogeneous) medium.
- The radiation to be monochromatic and unpolarized.

The complete specification of a prototype problem requires five input variables:

1. $\tau^*$, the vertical optical depth of the slab;
2. $S^*(\tau, \hat{\Omega})$, the internal or external sources;
3. $p(\hat{\Omega}', \hat{\Omega})$, the scattering phase function;
4. $\varpi$, the single-scattering albedo; and
5. $\rho(-\hat{\Omega}', \hat{\Omega})$, the bidirectional reflectance of the underlying surface ($\rho_{\rm L}$ = constant for a Lambert surface).

# Prototype Radiative Transfer Problems (2)

A cartoon illustration of the standard problems we describe below is provided in Fig. 3. The solution of the radiative transfer equation provides the following two sets of output variables:

1. the reflectance, transmittance, absorptance, and emittance; and
2. the source function, the internal radiance, the heating rate, and the net irradiance throughout the medium.

## Prototype Problem 1: Uniform Illumination

- The incident radiance has the same value $\mathcal{I}$ in all downward directions.
- Because of the azimuthal symmetry of the incident radiance, the radiation depends only on $\tau$ and $\mu$.
- Furthermore, the source function depends only upon $\tau$.

For conservative and isotropic scattering, the frequency-integrated problem reduces mathematically to that of a simple *greenhouse problem*.

- This problem describes the enhancement of the surface temperature over that expected for planetary radiative equilibrium as discussed in Chapter 8.

# Prototype Radiative Transfer Problems (3)

In addition, *Prototype Problem 1* approximately reproduces the illumination conditions provided by an optically thick cloud overlying an atmosphere. The source for the diffuse emission is

$$S^*(\tau) = \frac{\varpi}{4\pi} \int_0^{2\pi} d\phi \int_0^1 d\mu \mathcal{I} e^{-\tau/\mu} = \frac{\varpi}{2} \mathcal{I} E_2(\tau),$$

where $E_2(\tau)$ is the *second exponential integral* defined by Eq. 5.62. However,

- this problem is solved more efficiently by setting $S^* = 0$ in Eq. 1 and applying a uniform upper boundary condition to the *total* radiance $I^-(\tau = 0, \mu) = \mathcal{I}$.
- It is straightforward to add the effects of surface reflection, as described below.

## Prototype Problem 2: Constant Imbedded Source

- For thermal radiation problems, $(1 - \varpi)B$ "drives" of the scattered radiation.
- This "imbedded source" is a strong function of frequency and of course depends upon the temperature, through Eq. 4.4.
- In our *Prototype Problem 2*, we assume that the product $(1 - \varpi)B$ is constant with depth. In the conservative limit ($\varpi \to 1$), *Prototype Problem 2* is known as the *Milne Problem* (see Appendix S, §S.3.3).

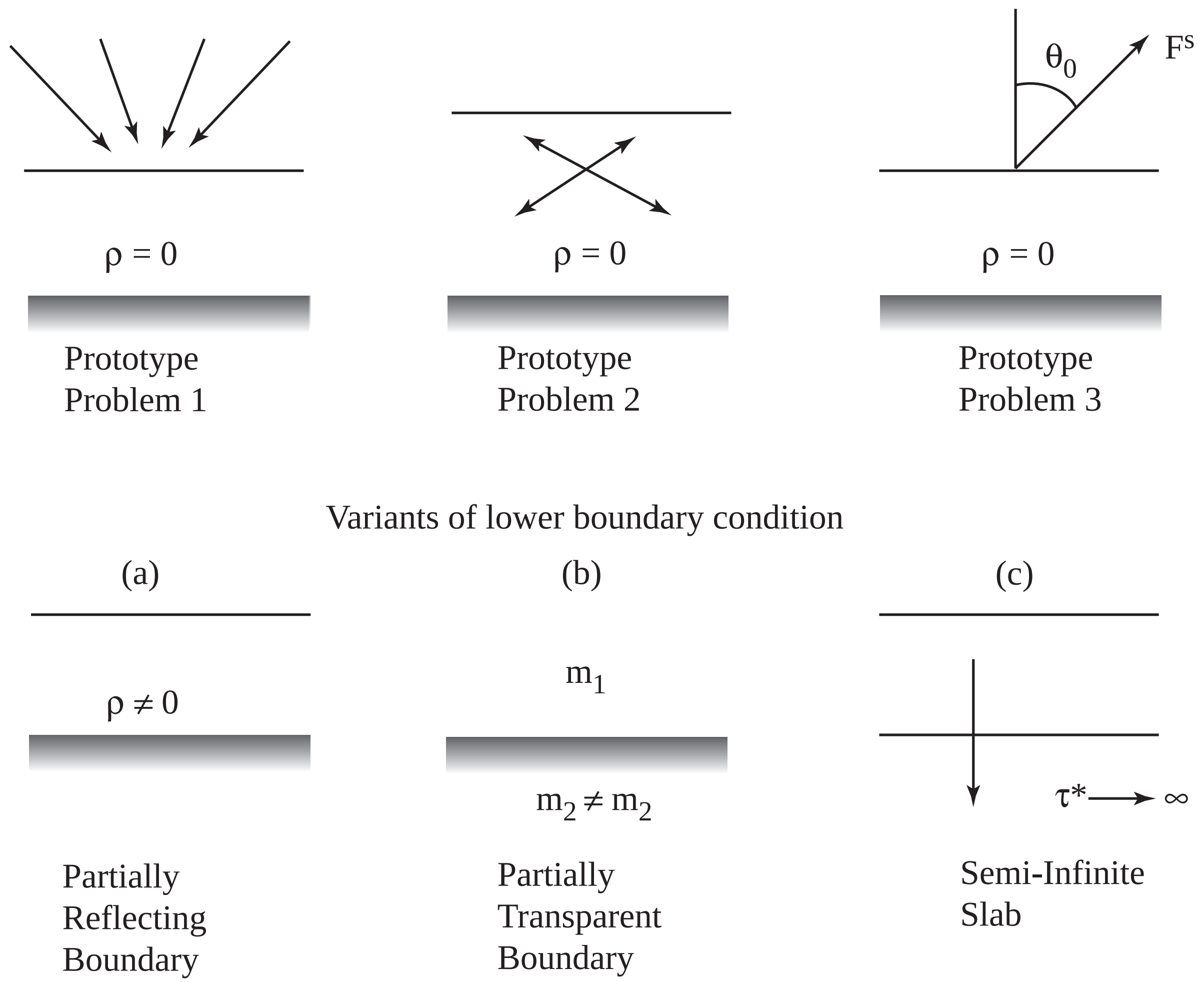


Figure 3: Illustration of Prototype Problems in radiative transfer.

# Prototype Radiative Transfer Problems (4)

## Prototype Problem 3: Diffuse Reflection Problem

In this problem, we consider collimated incidence at $\tau = 0$, and a lower boundary that may be partly reflecting as explained below.

- The case of collimated incidence as opposed to uniform incidence can be considered to be *the* classical planetary problem.
- For shortwave applications, the term $(1 - \varpi)B$ can be ignored and the only source term is

$$S^*(\tau, \pm\mu, \phi) = \frac{\varpi F^{\mathrm{s}}}{4\pi} p(-\mu_0, \phi_0; \pm\mu, \phi) e^{-\tau/\mu_0}. \tag{45}$$

- Note that in contrast to Prototype Problems 1 and 2, the radiation depends upon both $\mu$ and the azimuthal coordinate $\phi$. The lower boundary condition appropriate for this problem is described below.

## Boundary Conditions: Reflecting and Emitting Surface

We first consider a Lambertian surface (BRDF $= \rho_{\mathrm{L}}$), which also emits thermal IR radiation with an emittance $\epsilon_{\mathrm{s}}$ and temperature $T_{\mathrm{s}}$.

# Prototype Radiative Transfer Problems (5)

The upward radiance at the surface is given by (see Eq. 5.11)

$$I^+(\tau^*, \mu, \phi) = \rho_{\rm L} F_{\rm d}^-(\tau^*) + \rho_{\rm L} \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' I^-(0, \mu', \phi') e^{-\tau^*/\mu'} + \epsilon_{\rm s} B(T_{\rm s}). \quad (46)$$

$B(T_{\rm s})$ denotes the Planck function at the appropriate frequency, and $F_{\rm d}^-(\tau^*)$ is the downward *diffuse* irradiance at the lower boundary. The upper and lower boundary conditions for the three prototype problems can now be written down immediately:

*Prototype Problem 1*

$$I^-(0, \mu) = \mathcal{I}; \qquad I^+(\tau^*, \mu) = \rho_{\rm L}[F_{\rm d}^-(\tau^*) + 2\pi \mathcal{I} E_3(\tau^*)] + \epsilon_{\rm s} B(T_{\rm s}) \quad (47)$$

*Prototype Problem 2*

$$I^-(0, \mu) = 0; \qquad I^+(\tau^*, \mu) = \rho_{\rm L} F_{\rm d}^-(\tau^*) \quad (48)$$

*Prototype Problem 3*

$$I^-(0, \mu, \phi) = F^{\rm s}\delta(\mu - \mu_0)\delta(\phi - \phi_0); \; I^+(\tau^*, \mu, \phi) = \rho_{\rm L}[F_{\rm d}^-(\tau^*) + \mu_0 F^{\rm s} e^{-\tau^*/\mu_0}]. \quad (49)$$

The three equations above are expressed in terms of the unknown quantity $F_{\rm d}^-(\tau^*)$, which is not a difficulty for the methods of solution described in this book (Chapters 7 and 9).

# Prototype Radiative Transfer Problems (6)

For *Prototype Problem 3*, we may encounter situations where the surface is described by a more general reflectance condition, given by:

$$
\begin{aligned}
I^+(\tau^*, \mu, \phi) \;=\; & \int_0^{2\pi} d\phi' \int_0^1 d\mu' \mu' \rho_\mathrm{d}(-\mu', \phi'; \mu, \phi) I^-(\tau^*, +\mu', \phi') \\
& + \mu_0 F^\mathrm{s} e^{-\tau^*/\mu_0} \rho_\mathrm{d}(-\mu_0, \phi_0; +\mu, \phi) + \epsilon(\mu) B(T_\mathrm{s}),
\end{aligned}
\qquad (50)
$$

where we have assumed no $\phi$-dependence of the thermal emission, and $\rho_\mathrm{d}$ is the BRDF due to diffuse reflection.

Assuming that the BRDF depends only on the *difference* between the azimuthal angles of the incident and the reflected radiation, we may expand it as follows:

$$
\begin{aligned}
\rho_\mathrm{d}(-\mu', \phi'; \mu, \phi) \;&=\; \rho_\mathrm{d}(-\mu', \mu; \phi - \phi') \\
&=\; \sum_{m=0}^{2N-1} \rho_\mathrm{d}^m(-\mu', \mu) \cos m(\phi - \phi'),
\end{aligned}
\qquad (51)
$$

where the expansion coefficients are computed from

$$
\rho_\mathrm{d}^m(-\mu', \mu) \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} d(\phi - \phi') \rho_\mathrm{d}(-\mu', \mu; \phi - \phi') \cos m(\phi - \phi'). \qquad (52)
$$

# Prototype Radiative Transfer Problems (7)

Substituting Eq. 51 into Eq. 50 and using Eq. 35:

$$I(\tau, u, \phi) = \sum_{m=0}^{2N-1} I^m(\tau, u) \cos m(\phi_0 - \phi)$$

we find that each Fourier component must satisfy the bottom boundary condition (see Appendix O for a derivation):

$$\begin{aligned} I^{m+}(\tau^*, \mu) &= \delta_{m0}\epsilon(\mu)B(T_{\rm s}) + (1+\delta_{m0}) \int_0^1 d\mu' \mu' \rho_{\rm d}^m(-\mu', \mu) I^{m-}(\tau^*, \mu') \\ &+ \frac{\mu_0}{\pi} F^{\rm s} \rho_{\rm d}^m(-\mu_0, \mu) e^{-\tau^*/\mu_0} \qquad m = 0, 1, \cdots, 2N-1. \end{aligned} \tag{53}$$

Finally, for an atmosphere overlying a body of calm water, we must

- use the interface conditions provided in §6.4 (Eqs. 41–43) to account for the reflection and transmission taking place at the interface between the two strata with different refractive indices.
- Then a coupled atmosphere–water system should be considered as described in §6.4.

# Reciprocity, Duality, and Inhomogeneous Media (1)

As noted previously (see Appendices J and Q and Eq. 5.27) any satisfactory reflectance model must satisfy the **Reciprocity Principle**:

$$\rho(\theta', \phi'; \theta, \phi) = \rho(\theta, \phi; \theta', \phi').$$

Thus, the reciprocity relationships satisfied by the BRDF and the diffuse reflectance are:

$$\rho(-\hat{\Omega}', \hat{\Omega}) = \rho(-\hat{\Omega}, \hat{\Omega}'); \quad \rho(-\hat{\Omega}', 2\pi) = \rho(-2\pi, \hat{\Omega}') \tag{54}$$

where we have suppressed the $\nu$-argument.

The *Reciprocity Principle* states that in any linear system:

- the pathways leading from a cause (or action) at one point to an effect (or response) at another point can be traversed in the opposite direction.
- the BRDF (Eq. 54) is unchanged upon direction reversal of the light rays $\Rightarrow$
- the reflected radiance of *Prototype Problem* 1 is related to the diffuse reflectance of *Prototype Problem* 3 (see §6.7).

A relationship similar to Eq. (54) exists for the transmittance.

## Reciprocity, Duality, and Inhomogeneous Media (2)

Our discussion above applies only to **homogeneous** media:

- the optical properties, such as $\varpi$ and $p$, are uniform with optical depth.

When $\varpi$ and $p$ are NOT uniform with optical depth:

- the reflectance and transmittance for a slab illuminated from the **top** are in general different from those of the same slab illuminated from the **bottom**.

The reciprocity relationships for directional transmittance and hemispherical transmittance are:

$$\mathcal{T}(-\hat{\Omega}', -\hat{\Omega}) = \tilde{\mathcal{T}}(+\hat{\Omega}, +\hat{\Omega}'); \quad \mathcal{T}(-2\pi, -\hat{\Omega}) = \tilde{\mathcal{T}}(+\hat{\Omega}, +2\pi). \tag{55}$$

where we have denoted properties for illumination from below with the symbol $\tilde{}$.

- The principle of duality connects the transmittances of *Prototype Problems* 1 and 3, in a similar way as for reflectances.

The remaining relationships are:

$$\tilde{\rho}(+\hat{\Omega}', -\hat{\Omega}) = \tilde{\rho}(+\hat{\Omega}, -\hat{\Omega}'); \quad \tilde{\rho}(+\hat{\Omega}', -2\pi) = \tilde{\rho}(+2\pi, -\hat{\Omega}'). \tag{56}$$

# Reciprocity, Duality, and Inhomogeneous Media (3)

Practical implication: Suppose we are interested in:

- hemispherical reflectance and transmittance, and we want to solve a problem involving collimated radiation for many values of the incoming solar direction.

Then it is much more efficient to consider:

- the problem of uniform illumination and solve for the reflected radiance:

$$I^+(0,\hat{\Omega}) = \int_+ d\omega' \cos\theta' \rho(-\hat{\Omega}', +\hat{\Omega})\mathcal{I} = \mathcal{I}\rho(-2\pi, +\hat{\Omega}) = \mathcal{I}\rho(-\hat{\Omega}, +2\pi). \qquad (57)$$

The last result follows from Eq. 54. Thus:

- We can find $\rho(-\hat{\Omega}, +2\pi)$ for every value of $-\hat{\Omega}$ of interest, by applying uniform illumination with $\mathcal{I} = 1$ and solving for the radiance $I^+(0,\hat{\Omega})$.

Moreover, by integrating Eq. 57, we find:

$$F^+ = 2\pi \int_0^1 d\mu \mu I^+(0,\mu) = 2\pi\mathcal{I} \int_0^1 d\mu\mu\rho(-\mu, +2\pi) = \pi\mathcal{I}\bar{\rho}. \qquad (58)$$

Hence, we can obtain the spherical albedo, $\bar{\rho}$, by computing the diffuse reflectance, $F^+/\pi\mathcal{I}$, resulting from uniform illumination.

# Effects of Surface Reflection on the Radiation Field (1)

What are the effects of a reflecting lower boundary on the reflectance and transmittance of a homogeneous plane-parallel slab overlying a partially reflecting surface?

For Lambert surface we can:

- express the solutions for the emergent radiances, **the planetary problem**, algebraically in terms of the solutions for the **standard problem**, i.e., **a completely black or non-reflecting lower boundary.**

From Fig. 4, the total reflected irradiance from the combined slab plus lower boundary is the sum of the following components:

1. the reflection from the slab itself;
2. the irradiance that reaches the surface, is reflected, and then transmitted;
3. that part of (2) which is reflected back to the surface, and is then reflected a second time and transmitted;
4. all higher order terms, reflected three, four, or more times from the surface before being transmitted.

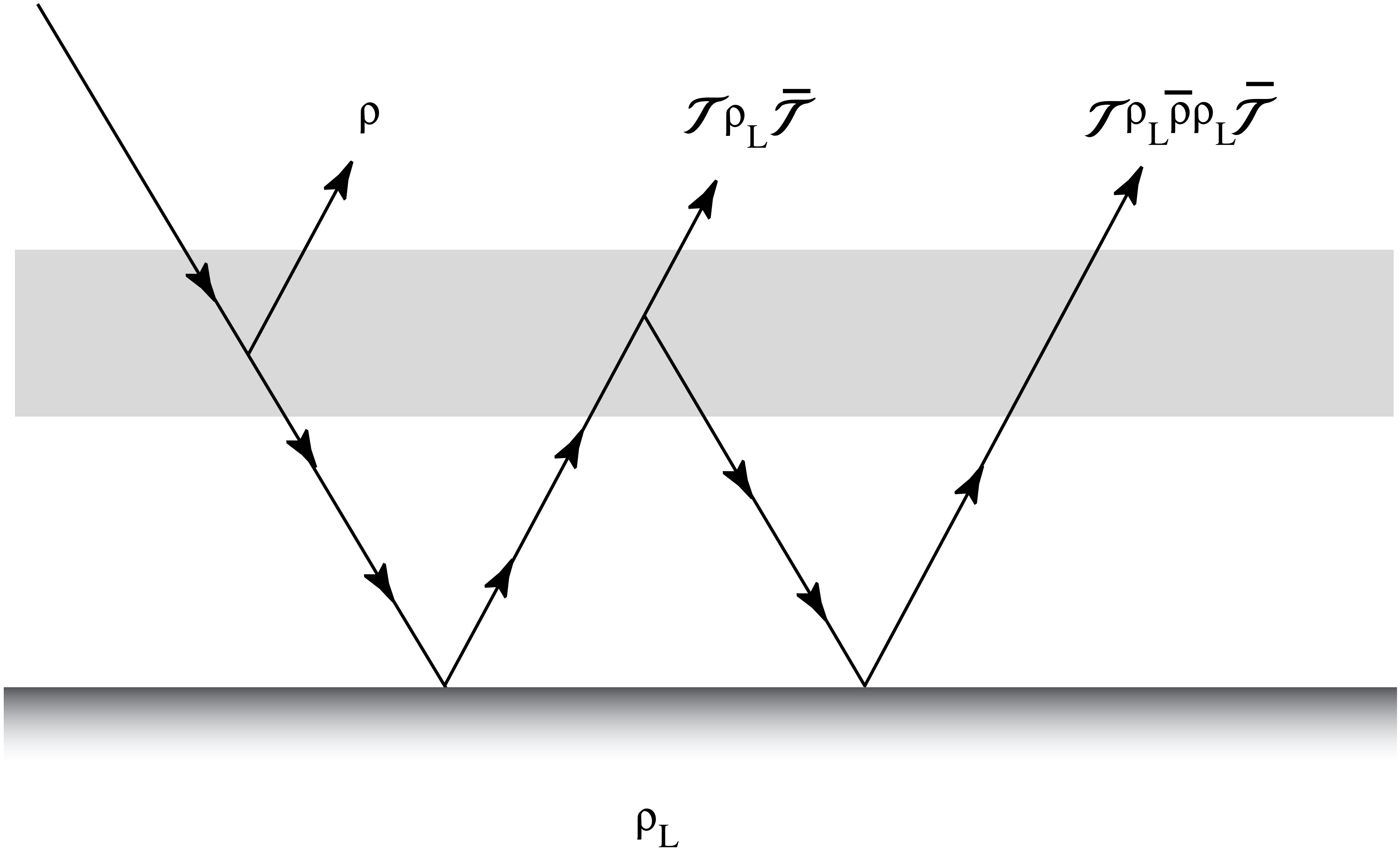


Figure 4: Addition of a reflecting surface leads to a geometric (binomial) series.

# Effects of Surface Reflection on the Radiation Field (3)

- The first term is just the ordinary hemispherical reflectance $\rho(-\hat{\Omega}, +2\pi)$.
- The second term is proportional to the hemispherical transmittance (including both the direct and diffuse components) $\mathcal{T}(-\hat{\Omega}, -2\pi)$. After reflection, it is proportional to $\mathcal{T}(-\hat{\Omega}, -2\pi)\rho_{\mathrm{L}}$, and upon transmission through the slab, it is multiplied by the spherical transmittance $\bar{\mathcal{T}}$. Thus:
- the second term is $\mathcal{T}(-\hat{\Omega}, -2\pi)\rho_{\mathrm{L}}\bar{\mathcal{T}}$.
- The third term takes the part of term (2) that was reflected from the surface, but instead of multiplying by the hemispherical transmittance, we multiply by the spherical reflectance $\bar{\rho}$, which "brings it back" for a second surface reflection. It is then multiplied by $\rho_{\mathrm{L}}$, and finally gets transmitted, bringing in the term $\bar{\mathcal{T}}$. Thus:
- the third term is $\mathcal{T}(-\hat{\Omega}, -2\pi)\rho_{\mathrm{L}}\bar{\rho}\rho_{\mathrm{L}}\bar{\mathcal{T}}$.

Proceeding in a similar way with the higher-order components, we find that the sum can be written as (see Fig. 4):

# Effects of Surface Reflection on the Radiation Field (4)

$$\rho(-\hat{\Omega},+2\pi)+\mathcal{T}(-\hat{\Omega},-2\pi)\rho_{\rm L}\bar{\mathcal{T}}\left[1+\bar{\rho}\rho_{\rm L}+(\bar{\rho}\rho_{\rm L})^2+(\bar{\rho}\rho_{\rm L})^3+\cdots\right]$$
$$=\ \rho(-\hat{\Omega},+2\pi)+\frac{\mathcal{T}(-\hat{\Omega},-2\pi)\rho_{\rm L}\bar{\mathcal{T}}}{1-\bar{\rho}\rho_{\rm L}}$$

where we have used the fact that the infinite sum is the binomial expansion of $(1-\bar{\rho}\rho_{\rm L})^{-1}$. Thus:

$$\rho_{\rm tot}(-\hat{\Omega},+2\pi,\rho_{\rm L})=\rho(-\hat{\Omega},+2\pi)+\frac{\mathcal{T}(-\hat{\Omega},-2\pi)\rho_{\rm L}\bar{\mathcal{T}}}{1-\bar{\rho}\rho_{\rm L}} \tag{59}$$

where the quantities on the right hand side ($\rho$, $\bar{\rho}$, $\mathcal{T}$, and $\bar{\mathcal{T}}$) are evaluated for a black surface, $\rho_{\rm L}=0$.

To include the possibility of **inhomogeneity** we replace $\bar{\rho}$ with $\tilde{\bar{\rho}}$, and $\bar{\mathcal{T}}$ with $\bar{\tilde{\bar{\mathcal{T}}}}$. Thus the total hemispherical reflectance is written:

$$\rho_{\rm tot}(-\hat{\Omega},+2\pi,\rho_{\rm L})=\rho(-\hat{\Omega},+2\pi)+\frac{\mathcal{T}(-\hat{\Omega},-2\pi)\rho_{\rm L}\bar{\tilde{\bar{\mathcal{T}}}}}{1-\tilde{\bar{\rho}}\rho_{\rm L}}. \tag{60}$$

# Effects of Surface Reflection on the Radiation Field (5)

The total hemispherical transmittance can be determined in a similar fashion:

$$\mathcal{T}_{\rm tot}(-\hat{\Omega}, -2\pi, \rho_{\rm L}) = \mathcal{T}(-\hat{\Omega}, -2\pi) + \frac{\mathcal{T}(-\hat{\Omega}, -2\pi)\rho_{\rm L}\tilde{\tilde{\bar{\rho}}}}{1 - \tilde{\tilde{\bar{\rho}}}\rho_{\rm L}} = \frac{\mathcal{T}(-\hat{\Omega}, -2\pi)}{1 - \tilde{\tilde{\bar{\rho}}}\rho_{\rm L}}. \qquad (61)$$

These results also follow from energy conservation.

We can derive similar relationships for the directional transmittance and reflectance (see Appendix Q for details):

$$\rho_{\rm tot}(-\hat{\Omega}_0, \hat{\Omega}, \rho_{\rm L}) = \rho(-\hat{\Omega}_0, \hat{\Omega}) + \frac{\rho_{\rm L}\mathcal{T}(-\hat{\Omega}_0, -2\pi)\tilde{\mathcal{T}}(\hat{\Omega}, +2\pi)}{\pi[1 - \tilde{\tilde{\bar{\rho}}}\rho_{\rm L}]} \qquad (62)$$

$$\mathcal{T}_{\rm tot}(-\hat{\Omega}_0, -\hat{\Omega}, \rho_{\rm L}) = \mathcal{T}(-\hat{\Omega}_0, -\hat{\Omega}) + \frac{\rho_{\rm L}\mathcal{T}(-\hat{\Omega}_0, -2\pi)\tilde{\rho}(\hat{\Omega}, +2\pi)}{\pi[1 - \tilde{\tilde{\bar{\rho}}}\rho_{\rm L}]}. \qquad (63)$$

We have shown that:

- the bidirectional reflectance and transmittance of a slab overlying a reflecting surface with a Lambertian reflectance is given by the sums and products of quantities **evaluated for a black surface**.

# Effects of Surface Reflection on the Radiation Field (6)

The implications are that:

- we need to solve only **one** radiative transfer problem involving a non-reflecting lower boundary.

BUT if the slab is **inhomogeneous** we should:

- apply uniform illumination from both the top and the bottom to allow for rapid computation of $\tilde{\mathcal{T}}(-\hat{\Omega}, -2\pi)$, $\tilde{\rho}(-\hat{\Omega}, +2\pi)$, $\bar{\tilde{\rho}}$ and $\bar{\tilde{\mathcal{T}}}$ as discussed above.

Thus:

- **as long as we are interested in only the transmitted and reflected radiances** an analytic correction allows us to find the solutions pertaining to reflecting (Lambert) surfaces.