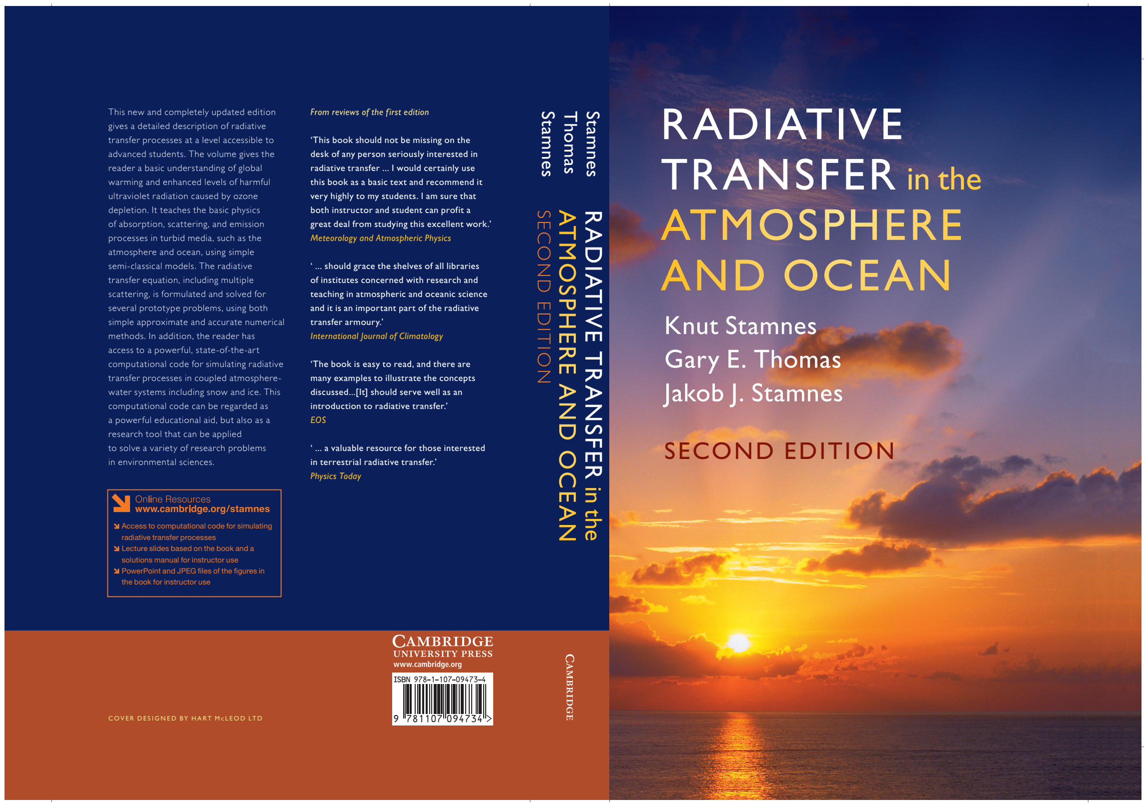
Lecture *Notes:*
Approximate Solutions of Prototype Problems



Based on Chapter 7 in K. Stamnes, G. E. Thomas, and J. J. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, 2017.

The Single Scattering Approximation (1)

In a slab geometry, the radiative transfer equation (RTE) for the full-range radiance is $(-1 \leq u \leq +1)$:

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \underbrace{\frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') - (1-\varpi)B - S^*(\tau,u,\phi)}_{S(\tau,u,\phi)} = I(\tau,u,\phi) - S(\tau,u,\phi).$$

We may integrate the RTE directly if the source function $S(\tau, u, \phi)$ is known.

- For thermal emission in the absence of scattering $S(\tau, u, \phi)$ is known.
- When multiple scattering is negligible $S(\tau, u, \phi)$ is known.
- The solution to the RTE in the absence of multiple scattering is usually referred to as the **single scattering approximation**.

In a slab geometry the RTEs for the half-range radiances are $(0 \leq \mu \leq 1)$:

$$\mu\frac{dI^+(\tau,\mu,\phi)}{d\tau} = I^+(\tau,\mu,\phi) - S^+(\tau,\mu,\phi) \tag{1}$$

$$-\mu\frac{dI^-(\tau,\mu,\phi)}{d\tau} = I^-(\tau,\mu,\phi) - S^-(\tau,\mu,\phi). \tag{2}$$

As usual, the independent variable is the optical depth τ, measured downwards from the "top" of the slab.

The Single Scattering Approximation (2)

We showed in Chapter 6 that formal solutions to these equations are a sum of direct ($I_{\rm s}$) and diffuse ($I_{\rm d}$) contributions:

$$
\begin{aligned}
I^-(\tau,\mu,\phi) &= \overbrace{I^-(0,\mu,\phi)e^{-\tau/\mu}}^{I_{\rm s}^-(\tau,\mu,\phi)} + \overbrace{\int_0^\tau \frac{d\tau'}{\mu} e^{-(\tau-\tau')/\mu} S^-(\tau',\mu,\phi)}^{I_{\rm d}^-(\tau,\mu,\phi)} && (3)\\
I^+(\tau,\mu,\phi) &= \overbrace{I^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu}}^{I_{\rm s}^+(\tau,\mu,\phi)} \\
&+ \overbrace{\int_\tau^{\tau^*} \frac{d\tau'}{\mu}\, e^{-(\tau'-\tau)/\mu} S^+(\tau',\mu,\phi)}^{I_d^+(\tau,\mu,\phi)}\,. && (4)\\
I(\tau,\mu=0,\phi) &= S(\tau). && (5)
\end{aligned}
$$

In the single scattering approximation, we assume that the multiple scattering contribution (the integral terms) to the source function is negligible.

The Single Scattering Approximation (3)

In the absence of multiple scattering, the source function simplifies as follows:

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \underbrace{\frac{\varpi}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{1} du' p(u',\phi';u,\phi) I(\tau,u',\phi') - (1-\varpi)B - S^*(\tau,u,\phi)}_{S(\tau,u,\phi)} \approx -(1-\varpi)B - S^*$$

$$\begin{aligned} S^{\pm}(\tau,\mu,\phi) &\approx (1-\varpi)B + S^*(\tau,\pm\mu,\phi) \\ &= (1-\varpi)B + \frac{\varpi F^{\mathrm{s}}}{4\pi} p(-\mu_0,\phi_0;\pm\mu,\phi)e^{-\tau/\mu_0}. \end{aligned} \tag{6}$$

- B is the Planck function.
- $(1-\varpi)$ the volume emittance.
- $S^{\pm}(\tau,\mu,\phi)$ are the **half-range source functions** analogous to the half-range **radiances** $I^{\pm}(\tau,\mu)$.
- Since $S^*(\tau,\pm\mu,\phi) = \frac{\varpi F^{\mathrm{s}}}{4\pi} p(-\mu_0,\phi_0;\pm\mu,\phi)e^{-\tau/\mu_0}$ – the contribution of singly-scattered solar radiation to the source function – varies exponentially with optical depth the integration can easily be carried out.

The Single Scattering Approximation (4)

Substituting the simplified form (Eq. 6) into Eqs. 3 and 4, and carrying out the integrations, we obtain the following analytic results for the first-order scattered (diffuse) radiance:

$$
\begin{aligned}
I_{\rm d}^{-}(\tau,\mu,\phi) \;&=\; (1-\varpi)B\left[1-e^{-\tau/\mu}\right] \\
&\quad + \frac{\varpi\mu_0 F^{\rm s} p(-\mu_0,\phi_0;-\mu,\phi)}{4\pi(\mu_0-\mu)}\left[e^{-\tau/\mu_0}-e^{-\tau/\mu}\right] \qquad (7)\\
I_{\rm d}^{+}(\tau,\mu,\phi) \;&=\; (1-\varpi)B\left[1-e^{-(\tau^*-\tau)/\mu}\right] \\
&\quad + \frac{\varpi\mu_0 F^{\rm s} p(-\mu_0,\phi_0;+\mu,\phi)}{4\pi(\mu_0+\mu)}\left[e^{-\tau/\mu_0}-e^{-[(\tau^*-\tau)/\mu+\tau^*/\mu_0]}\right]. \qquad (8)
\end{aligned}
$$

To obtain the total radiance, we must add to the diffuse radiance the boundary terms in Eqs. 3 and 4.

The Single Scattering Approximation (5)

Since $I^-(0, \mu, \phi) = F^{\mathrm{s}}\delta(\mu - \mu_0)\delta(\phi - \phi_0)$ and $I^+(\tau^*, \mu, \phi) = 0$ (if the lower boundary is assumed to be perfectly absorbing), the total radiance is given by:

$$\begin{aligned} I^-(\tau, \mu, \phi) &= I^-(0, \mu, \phi)e^{-\tau/\mu} + I_{\mathrm{d}}^-(\tau, \mu, \phi) \\ &= F^{\mathrm{s}} e^{-\tau/\mu}\delta(\mu - \mu_0)\delta(\phi - \phi_0) + I_{\mathrm{d}}^-(\tau, \mu, \phi) \qquad (9) \end{aligned}$$

$$I^+(\tau, \mu, \phi) = \overbrace{I^+(\tau^*, \mu, \phi)e^{-(\tau^*-\tau)/\mu}}^{0} + I_d^+(\tau, \mu, \phi) = I_d^+(\tau, \mu, \phi) \qquad (10)$$

where the diffuse terms $I_{\mathrm{d}}^-(\tau, \mu, \phi)$ and $I_{\mathrm{d}}^+(\tau, \mu, \phi)$ are given by Eqs. 7 and 8.

The Single Scattering Approximation (6)

Favorable aspects of the single-scattering approximation are:

1. The solution is valid for any scattering phase function.
2. It is easily generalized to include polarization.
3. It applies to any geometry, as long as we replace the slant optical depth τ/μ with the expression appropriate for the incident ray path. For example, in a spherical geometry, τ/μ_0 is replaced with $\tau Ch(\mu_0)$, where $Ch(\mu_0)$ is the *Chapman–Function*.
4. It is useful when an approximate solution is available for the multiple scattering. Then the diffuse radiance is given by the sum of single-scattering and (approximate) multiple-scattering contributions.
5. It serves as a starting point for expanding the radiation field in a sum of contributions from first-order scattering, second-order scattering, etc. The latter expansion technique, known as *Lambda iteration* (§7.2.2), allows us to evaluate more precisely the validity of the first-order scattering approximation. An alternative expansion technique is the *successive orders of scattering (SOS) method*, which is briefly discussed in §7.2.4.

The Two-Stream Approximation: Isotropic Scattering (1)

Approximate Differential Equations

The radiative transfer equations for the half-range radiances are given by (see Eqs. 6.3 and 6.4):

$$\mu\frac{dI^+(\tau,\mu)}{d\tau} = I^+(\tau,\mu) - \frac{\varpi}{2}\int_0^1 d\mu' I^+(\tau,\mu') - \frac{\varpi}{2}\int_0^1 d\mu' I^-(\tau,\mu') - (1-\varpi)B$$

$$-\mu\frac{dI^-(\tau,\mu)}{d\tau} = I^-(\tau,\mu) - \frac{\varpi}{2}\int_0^1 d\mu' I^+(\tau,\mu') - \frac{\varpi}{2}\int_0^1 d\mu' I^-(\tau,\mu') - (1-\varpi)B.$$

Because the scattering is isotropic, the radiation field has **no** azimuthal dependence as explained previously in Chapter 6.

- In the two-stream approximation, we replace the angularly-dependent quantities $I^{\pm}(\tau,\mu')$ by their averages over each hemisphere, $I^+(\tau)$ and $I^-(\tau)$ in each hemisphere: $I^{\pm}(\tau,\mu) \approx I^{\pm}(\tau)$.

The Two-Stream Approximation: Isotropic Scattering (2)

- This approximation leads to the following pair of coupled differential equations, which are called:

The two-stream equations

$$\bar{\mu}^{+}\frac{dI^{+}(\tau)}{d\tau} = I^{+}(\tau) - \frac{\varpi}{2}I^{+}(\tau) - \frac{\varpi}{2}I^{-}(\tau) - (1-\varpi)B \qquad (11)$$

$$-\bar{\mu}^{-}\frac{dI^{-}(\tau)}{d\tau} = I^{-}(\tau) - \frac{\varpi}{2}I^{+}(\tau) - \frac{\varpi}{2}I^{-}(\tau) - (1-\varpi)B. \qquad (12)$$

- Here $\bar{\mu}^{+}$ or $\bar{\mu}^{-}$ is the average of the cosine of the polar angle θ made by a beam in the upper or lower hemisphere, respectively.
- These linear, coupled, ordinary differential equations have simple solutions if the medium is homogeneous so that $\varpi(\tau) = \varpi =$ constant, and if also the Planck function B does not vary with τ: $B(\tau) = B =$ constant.

The Two-Stream Approximation: Isotropic Scattering (3)

Note that the two-stream approximation

- will be most accurate when the radiation field is nearly isotropic, i.e. deep inside the medium, far away from boundaries or sources or sinks of radiation, and it
- can teach us about radiative transfer in optically-thin as well as optically-thick media, both for scattering and emission-dominated problems.

The approximate two-stream expressions for the source function, the net irradiance, the mean radiance, and the heating rate are:

$$\begin{aligned} S(\tau) &= \frac{\varpi}{2}\int_0^1 d\mu[I^+(\tau,\mu)+I^-(\tau,\mu)]+(1-\varpi)B \\ &\approx \frac{\varpi}{2}\left[I^+(\tau)+I^-(\tau)\right]+(1-\varpi)B \end{aligned} \tag{13}$$

$$F(\tau) = 2\pi\int_0^1 d\mu\mu[I^+(\tau,\mu)-I^-(\tau,\mu)] \approx 2\pi\left[\bar{\mu}^+ I^+(\tau)-\bar{\mu}^- I^-(\tau)\right] \tag{14}$$

$$\bar{I}(\tau) = \frac{1}{4\pi}2\pi\int_0^1 d\mu[I^+(\tau,\mu)+I^-(\tau,\mu)] = \frac{1}{2}\left[I^+(\tau)+I^-(\tau)\right]$$

$$\mathcal{H}(\tau) = -\frac{\partial F}{\partial z} = 4\pi\alpha[\bar{I}(\tau)-B] \approx 2\pi\alpha\left[I^+(\tau)+I^-(\tau)\right]-4\pi\alpha B. \tag{15}$$

The Mean Inclination: Possible Choices for $\bar{\mu}$ (1)

In Eq. 15, α is the absorption coefficient. We have used the monochromatic version of Eq. 5.75 (Generalized Gershun's Law) for the heating rate.

- We could define $\bar{\mu}^{\pm}$ formally as the radiance-weighted angular means

$$\bar{\mu}^{\pm} = \langle\mu\rangle^{\pm} \equiv \frac{2\pi\int_0^1 d\mu\mu I^{\pm}(\tau,\mu)}{2\pi\int_0^1 d\mu I^{\pm}(\tau,\mu)} = \frac{F^{\pm}}{2\pi I^{\pm}}. \tag{16}$$

- But since we do not know the radiance distribution **a priori**, this definition is of little use, although it demonstrates that $\bar{\mu}^{+}$ and $\bar{\mu}^{-}$ will vary with optical depth and have different values.
- Hence, the common practise of picking the same constant value in both hemispheres ($\bar{\mu} = \bar{\mu}^{+} = \bar{\mu}^{-} = \text{constant}$) is clearly an approximation.
- If the radiance field were isotropic, then Eq. 16 yields $\bar{\mu}^{+} = \bar{\mu}^{-} = 1/2$ for all optical depths.
- If the radiance distribution were approximately linear in μ, say $I(\mu) \approx C\mu$, where C is a constant, then Eq. 16 yields $\bar{\mu} = 2/3$.

The Mean Inclination: Possible Choices for $\bar{\mu}$ (2)

- Alternatively, we could use the root-mean-square value:

$$\bar{\mu} \equiv \mu_{\rm rms} = \sqrt{\langle \mu^2 \rangle} = \sqrt{\frac{\int_0^1 d\mu \mu^2 I(\tau,\mu)}{\int_0^1 d\mu I(\tau,\mu)}}.$$

- If the radiation field were isotropic, this definition would yield $\bar{\mu} = 1/\sqrt{3}$.
- A linear variation of the radiation field would yield $\bar{\mu} = 1/\sqrt{2} = 0.71$.
- Thus, these possible choices yield $\bar{\mu}$–values ranging from 0.5 to 0.71, and there is no way to decide *a priori* which choice is optimal.
- We have to pick the optimal $\bar{\mu}$–value on **a trial-and-error basis** for each type of problem.
- We now assume a single value for $\bar{\mu}$ (the same value in both hemispheres) but leave its value undetermined to remind us that it represents some sort of average over a hemisphere.

Prototype Problem 1: Isotropic Incidence (1)

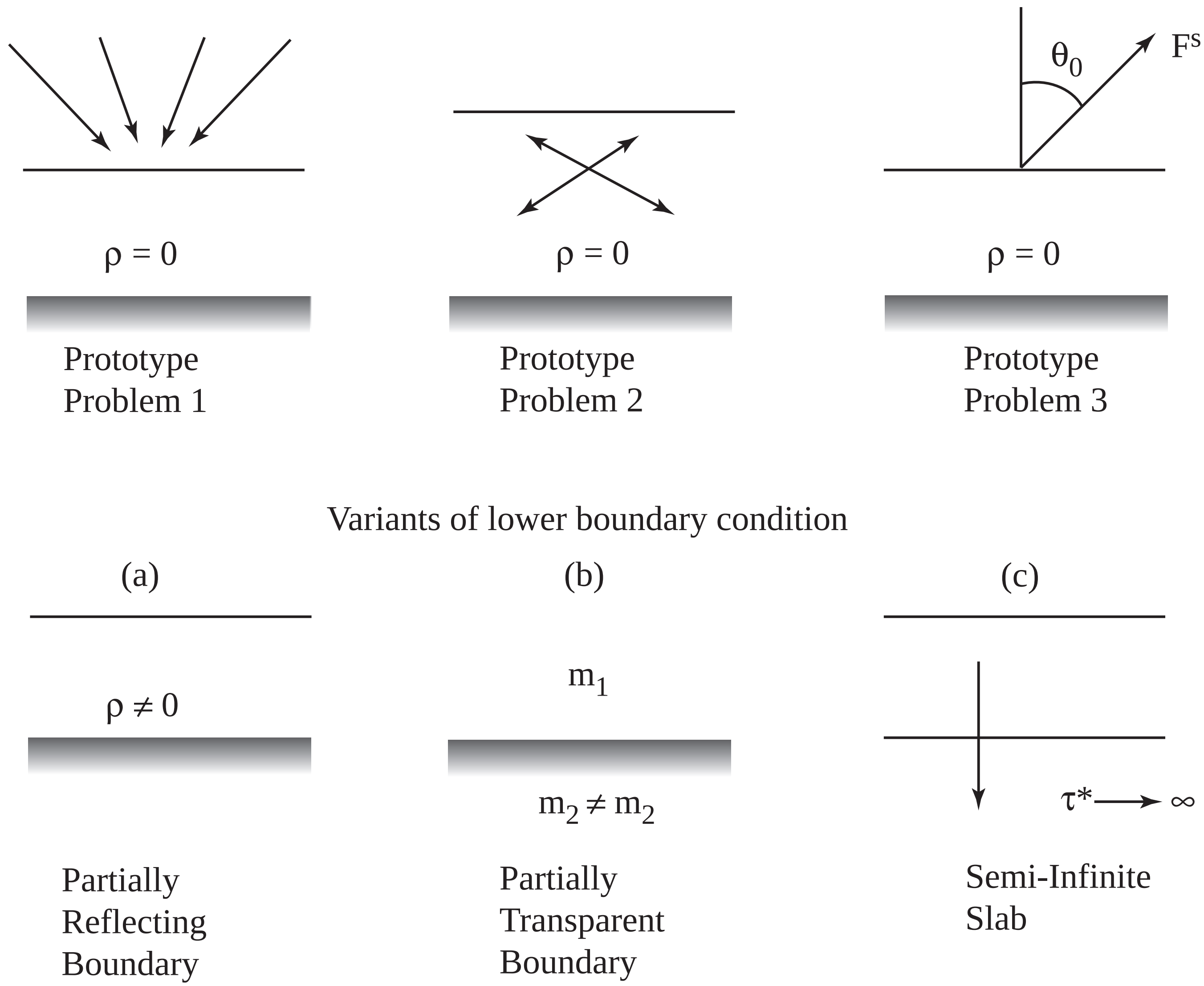


Figure 1: Illustration of Prototype Problems in radiative transfer.

Prototype Problem 1: Isotropic Incidence (2)

If we ignore thermal emission ($B = 0$), Eqs. 11 and 12 become (setting $\bar{\mu}^+ = \bar{\mu}^- = \bar{\mu}$):

$$\text{Eq. 11:} \quad \bar{\mu}\frac{dI^+(\tau)}{d\tau} = I^+(\tau) - \frac{\varpi}{2}I^+(\tau) - \frac{\varpi}{2}I^-(\tau)$$

$$\text{Eq. 12:} \quad -\bar{\mu}\frac{dI^-(\tau)}{d\tau} = I^-(\tau) - \frac{\varpi}{2}I^+(\tau) - \frac{\varpi}{2}I^-(\tau)$$

- By first adding Eqs. 11 and 12 and then subtracting Eq. 12 from Eq. 11, we obtain:

$$\bar{\mu}\frac{d(I^+ - I^-)}{d\tau} = (1 - \varpi)(I^+ + I^-) \qquad (17)$$

$$\bar{\mu}\frac{d(I^+ + I^-)}{d\tau} = (I^+ - I^-). \qquad (18)$$

- Differentiating Eq. 18 with respect to τ, and substituting for $d(I^+ - I^-)/d\tau$ from Eq. 17, we obtain a second-order differential equation involving only the **sum** of the radiances:

$$\frac{d^2(I^+ + I^-)}{d\tau^2} = \frac{(1 - \varpi)}{\bar{\mu}^2}(I^+ + I^-).$$

Prototype Problem 1: Isotropic Incidence (3)

- Similarly, differentiating Eq. 17 and substituting for $d(I^+ + I^-)/d\tau$ from Eq. 18, we obtain a second-order differential equation

$$\frac{d^2(I^+ - I^-)}{d\tau^2} = \frac{(1-\varpi)}{\bar{\mu}^2}(I^+ - I^-)$$

which involves only the **difference** between the two radiances.

- We have the same differential equation for the sum and the difference. Calling the unknown Y, we obtain a second-order **diffusion equation**

$$\frac{d^2 Y}{d\tau^2} = \Gamma^2 Y \qquad \text{where } \Gamma \equiv \sqrt{1-\varpi}/\bar{\mu} \tag{19}$$

for which the general solution is a sum of positive and negative exponentials

$$Y = A' e^{\Gamma\tau} + B' e^{-\Gamma\tau}$$

where A' and B' are integration constants to be determined.

Prototype Problem 1: Isotropic Incidence (4)

- Since the **sum** and **difference** of the two radiances are both expressed as sums of exponentials, each radiance component must be expressed in the same way:

$$I^+(\tau) = Ae^{\Gamma\tau} + Be^{-\Gamma\tau}; \qquad I^-(\tau) = Ce^{\Gamma\tau} + De^{-\Gamma\tau} \tag{20}$$

where A, B, C, and D are constants to be determined.

- We now introduce boundary conditions at the top and the bottom of the medium. For ***Prototype Problem 1***, we have:

$$I^-(\tau = 0) = \mathcal{I} = \text{ constant}; \qquad I^+(\tau^*) = 0. \tag{21}$$

- We choose this case of isotropic incidence as our first example, because the two-stream solution to this problem has a particularly simple form.

Prototype Problem 1: Isotropic Incidence (6)

- Equations 20:

$$I^+(\tau) = Ae^{\Gamma\tau} + Be^{-\Gamma\tau}; \qquad I^-(\tau) = Ce^{\Gamma\tau} + De^{-\Gamma\tau}$$

  have four integration constants, but the two boundary conditions in Eqs. 21: $I^-(\tau = 0) = \mathcal{I} =$ constant; $I^+(\tau^*) = 0$ for the second-order differential equation, suggest that there should be only two independent constants.

- To obtain the required relationships between A, B, C, and D, we substitute Eqs. 20 into Eqs. 11–12 to find that:

$$\frac{C}{A} = \frac{B}{D} = \frac{\varpi}{2 - \varpi + 2\bar{\mu}\Gamma} = \frac{1 - \bar{\mu}\Gamma}{1 + \bar{\mu}\Gamma} = \frac{1 - \sqrt{1 - \varpi}}{1 + \sqrt{1 - \varpi}} \equiv \rho_\infty. \tag{22}$$

- An explanation of the physical meaning of ρ_∞ is provided in Example 7.2 (see Eq. 39).

- Substitution from Eq. 22 into Eqs. 20 yields:

$$I^+(\tau) = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau} \tag{23}$$

$$I^-(\tau) = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}. \tag{24}$$

Prototype Problem 1: Isotropic Incidence (7)

- We now apply the boundary conditions in Eqs. 21, which yield:

$$I^-(\tau = 0) = \rho_\infty A + D = \mathcal{I} \qquad ; \qquad I^+(\tau = \tau^*) = Ae^{\Gamma\tau^*} + \rho_\infty De^{-\Gamma\tau^*} = 0.$$

- Solving for A and D we find

$$A = \frac{-\rho_\infty \mathcal{I} e^{-\Gamma\tau^*}}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}} \qquad ; \qquad D = \frac{\mathcal{I} e^{\Gamma\tau^*}}{e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}}.$$

- Hence, the solutions become

$$I^+(\tau) = \frac{\mathcal{I}\rho_\infty}{\mathcal{D}}\left[e^{\Gamma(\tau^*-\tau)} - e^{-\Gamma(\tau^*-\tau)}\right] \tag{25}$$

$$I^-(\tau) = \frac{\mathcal{I}}{\mathcal{D}}\left[e^{\Gamma(\tau^*-\tau)} - \rho_\infty^2 e^{-\Gamma(\tau^*-\tau)}\right] \tag{26}$$

where the denominator is:

$$\mathcal{D} \equiv e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}. \tag{27}$$

Prototype Problem 1: Isotropic Incidence (8)

The source function, irradiance and heating rate follow from Eqs. 13–15 ($\bar{\mu}^+ = \bar{\mu}^- = \bar{\mu}$):

$$S(\tau) = \frac{\varpi}{2}\int_0^1 d\mu[I^+(\tau,\mu) + I^-(\tau,\mu)] + (1-\varpi)B \approx \frac{\varpi}{2}\left[I^+(\tau) + I^-(\tau)\right] + (1-\varpi)B \quad \longleftarrow \quad \text{Eq. (13)}$$

$$F(\tau) = 2\pi\int_0^1 d\mu\mu[I^+(\tau,\mu) - I^-(\tau,\mu)] \approx 2\pi\bar{\mu}\left[I^+(\tau) - I^-(\tau)\right] \quad \longleftarrow \quad \text{Eq. (14)}$$

$$\mathcal{H}(\tau) = -\frac{\partial F}{\partial z} \approx 2\pi\alpha\left[I^+(\tau) + I^-(\tau)\right] - 4\pi\alpha B. \quad \longleftarrow \quad \text{Eq. (15)}$$

$$S(\tau) = \frac{\varpi\mathcal{I}}{2\mathcal{D}}(1+\rho_\infty)\left[e^{\Gamma(\tau^*-\tau)} - \rho_\infty e^{-\Gamma(\tau^*-\tau)}\right] \tag{28}$$

$$F(\tau) = -2\bar{\mu}\frac{\pi\mathcal{I}}{\mathcal{D}}(1-\rho_\infty)\left[e^{\Gamma(\tau^*-\tau)} + \rho_\infty e^{-\Gamma(\tau^*-\tau)}\right] \tag{29}$$

$$\mathcal{H}(\tau) = \frac{2\pi\alpha\mathcal{I}}{\mathcal{D}}(1+\rho_\infty)[e^{\Gamma(\tau^*-\tau)} - \rho_\infty e^{-\Gamma(\tau^*-\tau)}]. \tag{30}$$

- Note that Eq. 14 yields $F^-(0) = 2\pi\bar{\mu}I^-(0) = 2\pi\bar{\mu}\mathcal{I}$ at the top of the slab.
- We might be tempted to set $\bar{\mu} = 0.5$ so that this expression would yield the exact value $\pi\mathcal{I}$. However, to remain consistent with the two-stream approximation, we should use the approximate expression $2\pi\bar{\mu}$.

Prototype Problem 1: Isotropic Incidence (9)

The total reflectance, transmittance, and absorptance become:

$$\rho(-2\pi, 2\pi) = \frac{2\pi\bar{\mu}I^{+}(0)}{2\pi\bar{\mu}\mathcal{I}} = \frac{\rho_\infty}{\mathcal{D}}[e^{\Gamma\tau^*} - e^{-\Gamma\tau^*}] \tag{31}$$

$$\mathcal{T}(-2\pi, -2\pi) = 2\pi\bar{\mu}\frac{I^{-}(\tau^*)}{2\pi\bar{\mu}\mathcal{I}} = \frac{1-\rho_\infty^2}{\mathcal{D}} \tag{32}$$

$$\begin{aligned} \alpha(-2\pi) &= 1 - \rho(-2\pi, 2\pi) - \mathcal{T}(-2\pi, -2\pi) \\ &= \frac{(1-\rho_\infty)}{\mathcal{D}}\left[e^{\Gamma\tau^*} + \rho_\infty e^{-\Gamma\tau^*} - 1 - \rho_\infty\right]. \end{aligned} \tag{33}$$

Note that the total transmittance includes the "beam" transmittance:*

$$\mathcal{T}_{\rm b}(-2\pi, -2\pi) = \frac{\int_0^1 d\mu\mu\mathcal{I}e^{-\tau^*/\mu}}{\int_0^1 d\mu\mu\mathcal{I}} = 2E_3(\tau^*) \quad \longleftarrow \quad \textbf{beam transmittance}.$$

- Thus, the diffuse transmittance is:

$$\mathcal{T}_{\rm d}(-2\pi, -2\pi) = \mathcal{T}(-2\pi, -2\pi) - \mathcal{T}_{\rm b}(-2\pi, -2\pi) = \frac{1-\rho_\infty^2}{\mathcal{D}} - 2E_3(\tau^*). \tag{34}$$

$^*E_n(\tau) \equiv \int_0^1 d\mu\mu^{n-2}e^{-\tau/\mu}$

Example: Semi-infinite slab (1)

- The limit of $\tau^* \to \infty$ is an approximation to a very thick atmosphere (such as Venus or Jupiter) or a deep ocean. Invoking the condition $S(\tau)e^{-\tau} \to 0$, we must exclude the positive-exponential terms. The solutions simplify as follows:

$$
\begin{aligned}
I^-(\tau) &= \mathcal{I}e^{-\Gamma\tau} \qquad ; \qquad I^+(\tau) = \mathcal{I}\rho_\infty e^{-\Gamma\tau} && (35)\\
S(\tau) &= \frac{\varpi}{2}\mathcal{I}(1+\rho_\infty)e^{-\Gamma\tau} && (36)\\
F(\tau) &= -2\pi\bar{\mu}\mathcal{I}(1-\rho_\infty)e^{-\Gamma\tau} && (37)\\
\mathcal{H}(\tau) &= \frac{\alpha\mathcal{I}(1+\rho_\infty)}{2\bar{\mu}}e^{-\Gamma\tau}. && (38)
\end{aligned}
$$

Note that:

- the sign of $F(\tau)$ is negative, indicating that the net flow of energy is downward.

- $F(\tau) \to 0$ as $\Gamma\tau >> 1$, or as $\tau >> 1/\Gamma$, i.e. as τ exceeds many thermalization lengths, Γ^{-1}, defined below.

Example: Semi-infinite slab (2)

- The diffuse reflectance of the semi-infinite slab is:

$$\rho(-2\pi, 2\pi) = \frac{I^+(\tau = 0)}{\mathcal{I}} = \frac{\mathcal{I}\rho_\infty}{\mathcal{I}} = \rho_\infty = \frac{1 - \sqrt{1-\varpi}}{1 + \sqrt{1-\varpi}} \tag{39}$$

- which explains the meaning of the notation ρ_∞. This two-stream approximation result for the diffuse reflectance turns out to be the exact result for the reflected radiance (see Exercise 7.7).
- Eq. 39 also provides us with the absorptance:

$$\alpha(-2\pi) = 1 - \rho(-2\pi, 2\pi) = \frac{2\sqrt{1-\varpi}}{1 + \sqrt{1-\varpi}}. \tag{40}$$

Example: Thermalization Length and Random Walk (1)

- Equation 35 and Eqs. 25 and 26 show that the $(1/e)$-depth of photon penetration is:

$$\Gamma^{-1} = \frac{\bar{\mu}}{\sqrt{1-\varpi}}$$

- which is called the **thermalization length**. It is interpreted to be the mean optical depth of photon penetration after repeated scatterings before absorption (thermalization).
- The dependence on $\bar{\mu}$ is straightforward, since the steeper the mean inclination of the rays (i.e. the smaller the value of $\bar{\mu}$), the smaller the penetration.
- The dependence on $1/\sqrt{1-\varpi}$ is qualitatively reasonable; when $\varpi \to 1$ (conservative scattering or no absorption), the penetration depth becomes infinite, as one would expect in the case of no absorption.

Example: Thermalization Length and Random Walk (2)

- But why should the *square root* of the volume emittance $(1-\varpi)$ be the relevant dependence?
- To answer this question we will use the **random walk** picture of "diffusion" of a multiply-scattered photon over distance.
- Imagine that photons are released repeatedly into an unbounded scattering medium, and are randomly scattered, on the average $\langle N \rangle$ times before being destroyed (absorbed). Since the probability of being destroyed *per collision* is $(1-\varpi)$, then it is clear that $\langle N \rangle(1-\varpi) = 1 \;\Rightarrow\; \langle N \rangle = 1/(1-\varpi)$.
- According to the random-walk theory, the mean total distance through which an average photon "wanders" after $\langle N \rangle$ collisions is $\sqrt{\langle N \rangle} l_{\rm mfp}$, where $l_{\rm mfp}$ is the photon mean free path.
- Now $l_{\rm mfp}$ is just one optical depth times the cosine of the mean ray inclination, $l_{\rm mfp} = \bar{\mu}$. Thus, the mean total distance covered before being absorbed is just

$$\sqrt{\langle N \rangle} \cdot \bar{\mu} = \bar{\mu}/\sqrt{1-\varpi}.$$

Example: The Conservative-Scattering Limit (1)

There are two ways to find this solution:

- The first is to take the limit $\varpi \to 1$ of the expressions valid for non-conservative scattering using *L'Hôpital's Rule* to handle the 0/0 limits.
- The second way is to return to the (simplified) set of coupled differential equations, and solve them afresh. Then it turns out that only a *first-order* differential equation must be solved. Both methods yield the following results (see Exercise 7.2).

$$I^+(\tau) = \frac{\mathcal{I}(\tau^* - \tau)}{2\bar{\mu} + \tau^*}; \qquad I^-(\tau) = \frac{\mathcal{I}[2\bar{\mu} + (\tau^* - \tau)]}{2\bar{\mu} + \tau^*} \tag{41}$$

$$S(\tau) = \frac{\mathcal{I}[\bar{\mu} + (\tau^* - \tau)]}{2\bar{\mu} + \tau^*}; \qquad F(\tau) = -\frac{4\pi\bar{\mu}^2\mathcal{I}}{2\bar{\mu} + \tau^*}; \qquad \mathcal{H}(\tau) = 0. \tag{42}$$

- Note that the irradiance is constant throughout the medium and the heating rate is zero, as expected for a conservatively-scattering slab (no absorption).

Example: The Conservative-Scattering Limit (2)

- These results were derived by Schuster in 1905, one of the first published solutions of the radiative transfer equation.
- Assuming $\bar{\mu} = 1/2$, Schuster found the diffuse reflectance and transmittance of the medium to be:

$$\boxed{\rho(-2\pi, 2\pi) = \frac{I^+(0)}{\mathcal{I}} = \frac{\tau^*}{1+\tau^*}; \qquad \mathcal{T}(-2\pi, -2\pi) = \frac{I^-(\tau^*)}{\mathcal{I}} = \frac{1}{1+\tau^*}.}$$

- Schuster also pointed out the following remarkable property of a conservatively-scattering slab:
- For $\tau^* \gg 1$, its transmittance is inversely proportional to the optical thickness τ^*.
- In contrast, for an absorbing slab (with no scattering) the transmittance is $e^{-\tau^*}$.

Example: Angular Distribution of the Radiation Field (1)

- To find the radiance $I^{\pm}(\tau, \mu)$ in the two-stream approximation at arbitrary values of μ, it is necessary to integrate the (approximate) source function.
- This method yields a closed-form solution for the angular dependence of the radiance, and may provide sufficient accuracy for some problems.
- We proceed by considering the expressions for the upward and downward radiance:

$$
\begin{aligned}
I^+(\tau, \mu) &= \int_\tau^{\tau^*} \frac{d\tau'}{\mu} S(\tau') e^{-(\tau'-\tau)/\mu} \\
I^-(\tau, \mu) &= \int_0^{\tau} \frac{d\tau'}{\mu} S(\tau') e^{-(\tau-\tau')/\mu} + \mathcal{I} e^{-\tau/\mu}.
\end{aligned} \tag{43}
$$

Example: Angular Distribution of the Radiation Field (2)

- Inserting the approximate two-stream source function:

$$S(\tau) = \frac{\varpi\mathcal{I}}{2\mathcal{D}}(1+\rho_\infty)\left[e^{\Gamma(\tau^*-\tau)} - \rho_\infty e^{-\Gamma(\tau^*-\tau)}\right]$$

into Eq. 43 and performing the integration, we find:

$$\begin{aligned} I^+(\tau,\mu) \;&=\; \frac{\mathcal{I}\rho_\infty}{\mathcal{D}}\{C^+(\mu)e^{\Gamma(\tau^*-\tau)} - C^-(\mu)e^{-\Gamma(\tau^*-\tau)} \\ &\quad + [C^-(\mu) - C^+(\mu)]e^{-(\tau^*-\tau)/\mu}\} \end{aligned} \tag{44}$$

$$\begin{aligned} I^-(\tau,\mu) \;&=\; \frac{\mathcal{I}}{\mathcal{D}}\{C^-(\mu)e^{\Gamma(\tau^*-\tau)} - C^+(\mu)\rho_\infty^2 e^{-\Gamma(\tau^*-\tau)} + [1 - C^-(\mu)]e^{\Gamma\tau^*-\tau/\mu} \\ &\quad - \rho_\infty^2[1 - C^+(\mu)]e^{-\Gamma\tau^*-\tau/\mu}\} \end{aligned} \tag{45}$$

where

$$C^\pm(\mu) \equiv (1 \pm \Gamma\bar{\mu})/(1 \pm \Gamma\mu).$$

Example: Angular Distribution of the Radiation Field (3)

- This form is convenient because it shows explicitly that when $\mu = \bar{\mu}$, then $C^{\pm}(\bar{\mu}) = 1$ and the results become identical to Eqs. 25–26:

$$I^+(\tau) = \frac{\mathcal{I}\rho_\infty}{\mathcal{D}}\left[e^{\Gamma(\tau^*-\tau)} - e^{-\Gamma(\tau^*-\tau)}\right] \tag{46}$$

$$I^-(\tau) = \frac{\mathcal{I}}{\mathcal{D}}\left[e^{\Gamma(\tau^*-\tau)} - \rho_\infty^2 e^{-\Gamma(\tau^*-\tau)}\right]. \tag{47}$$

- For horizontal viewing ($\mu = 0$), Eqs. 44 and 45 approach the source function, $I^{\pm}(\tau, \mu \to 0) = S(\tau)$, which is a property shared by the exact result.
- It is easily verified that the above results satisfy the boundary conditions for all values of μ, that is, $I^-(0,\mu) = \mathcal{I}$ and $I^+(\tau^*, \mu) = 0$.
- The expressions for the hemispherical-directional reflectance $I^+(0,\mu)/\mathcal{I}$ and the hemispherical-directional transmittance $I^-(\tau^*, \mu)/\mathcal{I}$ become:

Example: Angular Distribution of the Radiation Field (4)

$$\rho(-2\pi,\mu) = \frac{\rho_\infty}{\mathcal{D}}\{C^+(\mu)e^{\Gamma\tau^*} - C^-(\mu)e^{-\Gamma\tau^*} + [C^-(\mu) - C^+(\mu)]e^{-\tau^*/\mu}\} \tag{48}$$

$$\begin{aligned}\mathcal{T}(-2\pi,-\mu) &= \frac{1}{\mathcal{D}}\{C^-(\mu) - \rho_\infty^2 C^+(\mu) + [1 - C^-(\mu)]e^{\Gamma\tau^* - \tau^*/\mu} \\ &\quad - \rho_\infty^2[1 - C^+(\mu)]e^{-\Gamma\tau^* - \tau^*/\mu}\}.\end{aligned} \tag{49}$$

In the special case $\mu = \bar{\mu}$, we have $C^\pm(\bar{\mu}) = 1$, and the diffuse reflectance (Eq. 48) agree with the two-stream result (Eq. 31):

$$\rho(-2\pi,\bar{\mu}) = \rho(-2\pi,2\pi) = \frac{2\pi\bar{\mu}I^+(0)}{2\pi\bar{\mu}\mathcal{I}} = \frac{\rho_\infty}{\mathcal{D}}[e^{\Gamma\tau^*} - e^{-\Gamma\tau^*}]$$

and the diffuse transmittance (Eq. 49) agree with the two-stream result (Eq. 32):

$$\mathcal{T}(-2\pi,-\bar{\mu}) = \mathcal{T}(-2\pi,-2\pi) = 2\pi\bar{\mu}\frac{I^-(\tau^*)}{2\pi\bar{\mu}\mathcal{I}} = \frac{1-\rho_\infty^2}{\mathcal{D}}.$$

According to the duality principle, these results are equal to the **directional**-hemispherical reflectance and the **directional**-hemispherical transmittance, i.e. $\rho(-\mu,2\pi) = \rho(-2\pi,\mu)$ and $\mathcal{T}(-\mu,-2\pi) = \mathcal{T}(-2\pi,-\mu)$.

Prototype Problem 2: Internal Source (1)

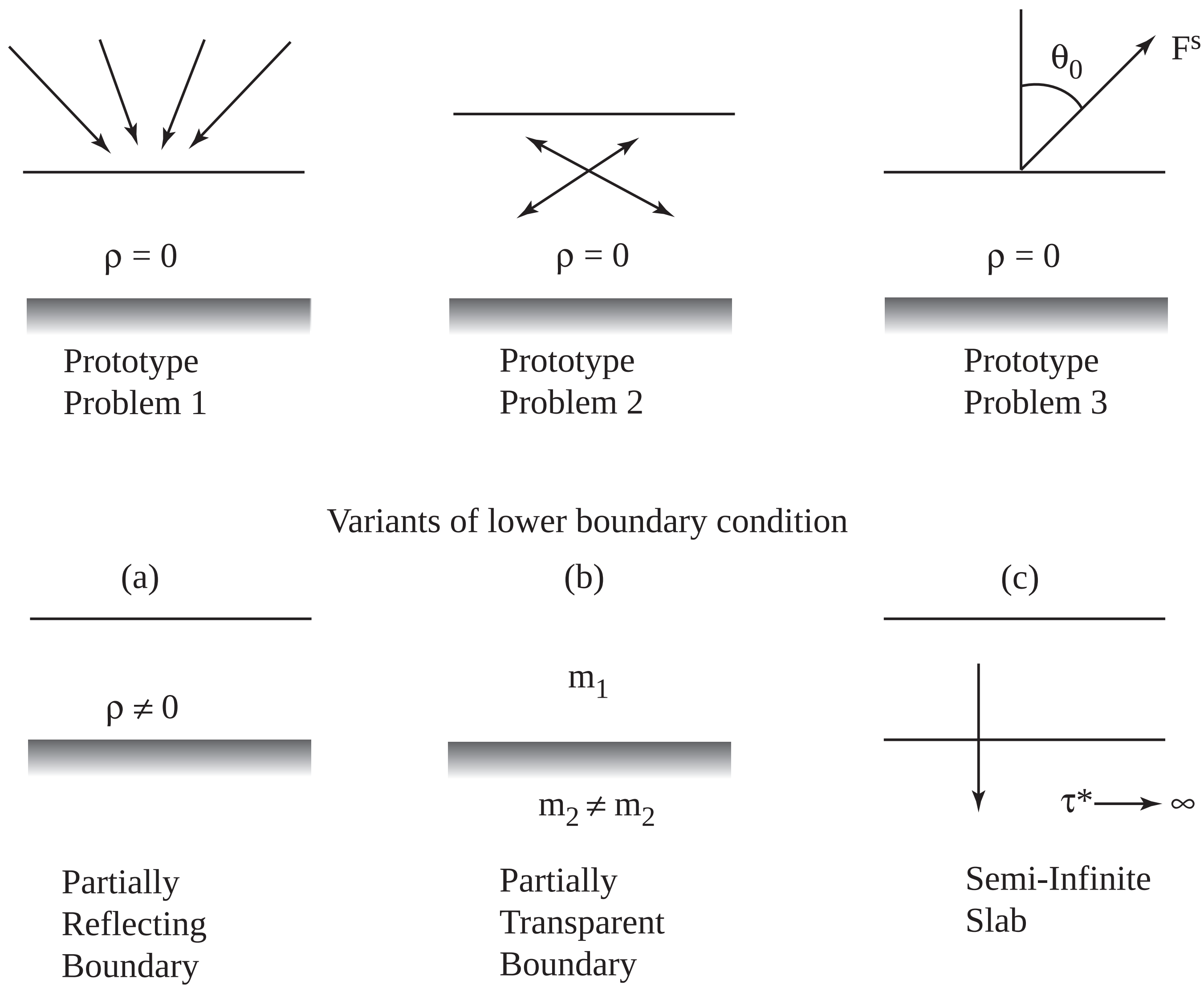


Figure 2: Illustration of Prototype Problems in radiative transfer.

Prototype Problem 2: Internal Source (2)

- Consider now *Prototype Problem 2*, where the only source of radiation is thermal emission within the slab.
- We assume that the slab is isothermal, isotropically-scattering, and homogeneous.

The two-stream equations are:

$$\bar{\mu}\frac{dI^+(\tau)}{d\tau} = I^+(\tau) - \frac{\varpi}{2}I^+(\tau) - \frac{\varpi}{2}I^-(\tau) - \underbrace{(1-\varpi)B}_{internal\ source} \tag{50}$$

$$-\bar{\mu}\frac{dI^-(\tau)}{d\tau} = I^-(\tau) - \frac{\varpi}{2}I^+(\tau) - \frac{\varpi}{2}I^-(\tau) - \underbrace{(1-\varpi)B}_{internal\ source} \tag{51}$$

with the boundary conditions $I^-(0) = I^+(\tau^*) = 0$.

- These equations differ from the previous set by having an extra **inhomogeneous term** $(1-\varpi)B$**,** on the RHS.

Prototype Problem 2: Internal Source (3)

- It is standard practice to first seek a solution to the **homogeneous equation** for which the imbedded source $(1-\varpi)B$ is set equal to zero. Next, we find a **particular solution** which satisfies the full equation.
- The general solution is then the sum of the homogeneous and particular solutions.

We seek homogeneous solutions of the form derived previously:

$$I_h^+(\tau) = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau}; \qquad I_h^-(\tau) = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}$$

where

- A and B are to be determined from the boundary conditions, and
- Γ and ρ_∞ are as defined previously:

$$\Gamma \equiv \sqrt{1-\varpi}/\bar{\mu}$$

$$\rho_\infty = \frac{1-\sqrt{1-\varpi}}{1+\sqrt{1-\varpi}}.$$

Prototype Problem 2: Internal Source (4)

- The particular solution is obtained by guessing that $I_{\rm p}^{+} = B$ and $I_{\rm p}^{-} = B$ are solutions, which is easily verified by substituting these guess solutions into the governing equations. Hence the complete solutions become: $I_{\rm tot}^{\pm} = I_{\rm h}^{\pm} + I_{\rm p}^{\pm}$:

$$I_{\rm tot}^{+}(\tau) = Ae^{\Gamma\tau} + \rho_{\infty} De^{-\Gamma\tau} + B; \qquad I_{\rm tot}^{-}(\tau) = \rho_{\infty} Ae^{\Gamma\tau} + De^{-\Gamma\tau} + B.$$

- Imposing the boundary conditions: $I^{-}(0) = I^{+}(\tau^{*}) = 0$, we arrive at the following two equations:

$$Ae^{\Gamma\tau^{*}} + \rho_{\infty} De^{-\Gamma\tau^{*}} + B = 0 \qquad \rho_{\infty} A + D + B = 0$$

which solved for A and D yields:

$$\boxed{A = \frac{-B(1 - \rho_{\infty} e^{-\Gamma\tau^{*}})}{\mathcal{D}}; \qquad D = \frac{-B(e^{\Gamma\tau^{*}} - \rho_{\infty})}{\mathcal{D}}}$$

with as defined previously (Eq. 27):

$$\mathcal{D} \equiv e^{\Gamma\tau^{*}} - \rho_{\infty}^{2} e^{-\Gamma\tau^{*}}.$$

Prototype Problem 2: Internal Source (5)

- Substituting these results into the general solution, we find:

$$I^+(\tau) = \frac{B}{\mathcal{D}}\{\rho_\infty^2 e^{-\Gamma\tau} - e^{\Gamma\tau} + \rho_\infty[e^{-\Gamma(\tau^*-\tau)} - e^{\Gamma(\tau^*-\tau)}]\} + B \tag{52}$$

$$I^-(\tau) = \frac{B}{\mathcal{D}}\{\rho_\infty^2 e^{-\Gamma(\tau^*-\tau)} - e^{\Gamma(\tau^*-\tau)} + \rho_\infty[e^{-\Gamma\tau} - e^{\Gamma\tau}]\} + B. \tag{53}$$

The expression for the irradiance is, from Eq. 14: $F(\tau) = 2\pi\bar{\mu}\left[I^+(\tau) - I^-(\tau)\right]$:

$$F(\tau) = 2\pi\bar{\mu}\frac{B}{\mathcal{D}}\left\{\rho_\infty^2\left[e^{-\Gamma\tau} - e^{-\Gamma(\tau^*-\tau)}\right] + \left[e^{\Gamma(\tau^*-\tau)} - e^{\Gamma\tau}\right]\right\}$$

$$+2\pi\bar{\mu}\frac{B}{\mathcal{D}}\rho_\infty[e^{-\Gamma(\tau^*-\tau)} - e^{-\Gamma\tau} - e^{\Gamma(\tau^*-\tau)} + e^{\Gamma\tau}] \tag{54}$$

and the source function is, from Eq. 13:

$$S(\tau) = \frac{\varpi}{2}\left[I^+(\tau) + I^-(\tau)\right] + (1-\varpi)B$$

which yields:

Prototype Problem 2: Internal Source (6)

$$S(\tau) = \frac{\varpi}{2}(I^+ + I^-) + (1-\varpi)B \tag{55}$$

$$\frac{S(\tau)}{B} = 1 - \frac{\varpi(1+\rho_\infty)}{2\mathcal{D}}[e^{\Gamma\tau} - \rho_\infty e^{-\Gamma(\tau^*-\tau)} + e^{\Gamma(\tau^*-\tau)} - \rho_\infty e^{-\Gamma\tau}]. \tag{56}$$

- The slab emittance, or bulk emittance is:

$$\epsilon(2\pi) = \frac{I^+(0)}{B} = \frac{I^-(\tau^*)}{B} = \frac{1}{\mathcal{D}}[\rho_\infty^2 - 1 - \rho_\infty(e^{\Gamma\tau^*} - e^{-\Gamma\tau^*})] + 1. \tag{57}$$

- For a discussion of the accuracy of the two-stream approximation, see §S.4.5 of Appendix S.

Prototype Problem 3: Beam Incidence (1)

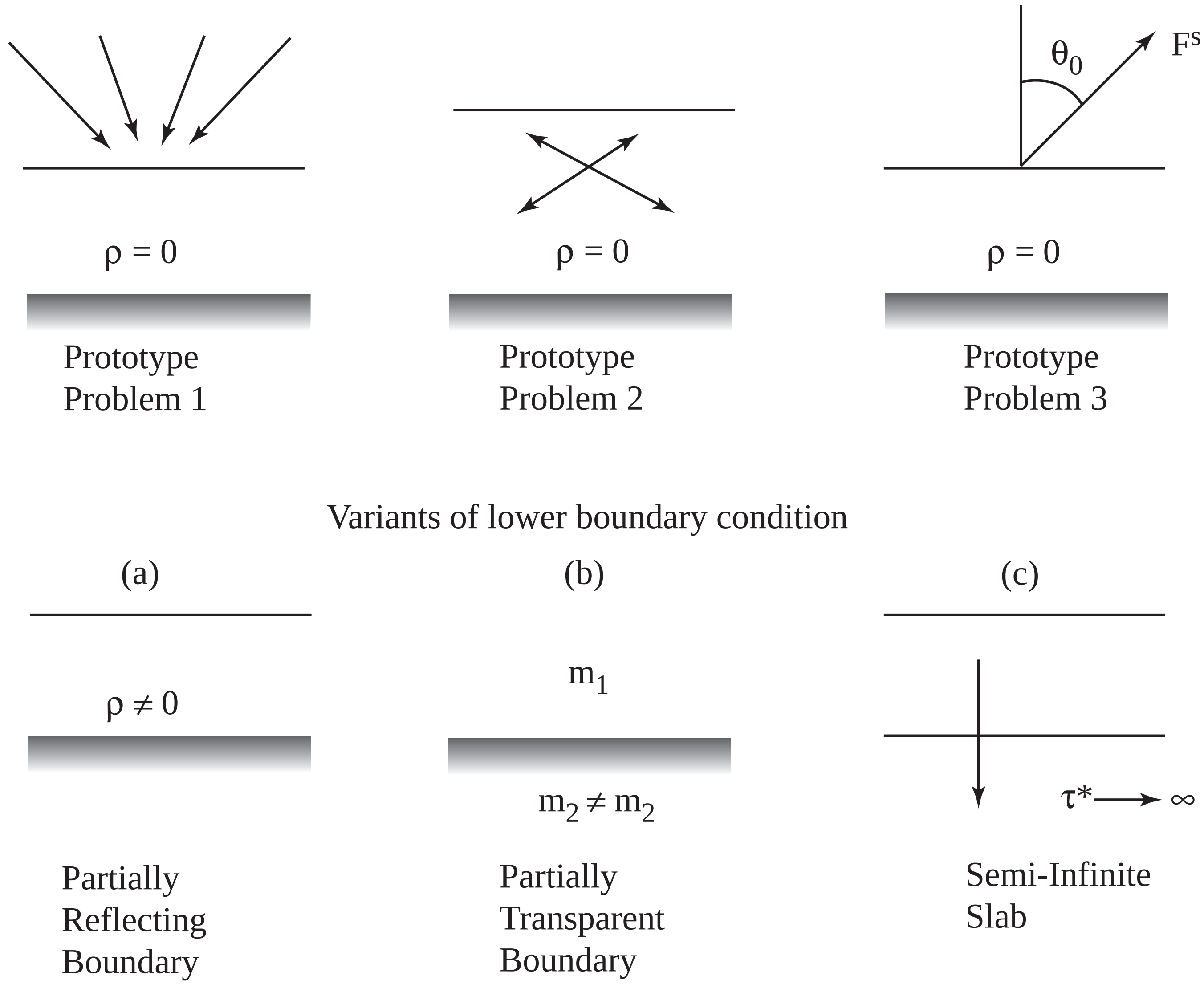


Figure 3: Illustration of Prototype Problems in radiative transfer.

Prototype Problem 3: Beam Incidence (2)

- We now consider the most important scattering problem in planetary atmospheres – that of a collimated solar beam of irradiance F^{s}, incident from above on a planetary atmosphere.
- We simplify to an isotropically-scattering, homogeneous atmosphere and, as usual, assume a black lower boundary. (Both these restrictions will be removed later.)

Setting the angle of incidence to be $\theta_0 = \cos^{-1}\mu_0$, we find that the appropriate two-stream equations are:

$$\bar{\mu}\frac{dI_{\mathrm{d}}^{+}}{d\tau} = I_{d}^{+} - \frac{\varpi}{2}(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-}) - \underbrace{\frac{\varpi}{4\pi}F^{\mathrm{s}}e^{-\tau/\mu_0}}_{solar\ pseudo-source} \tag{58}$$

$$-\bar{\mu}\frac{dI_{\mathrm{d}}^{-}}{d\tau} = I_{\mathrm{d}}^{-} - \frac{\varpi}{2}(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-}) - \underbrace{\frac{\varpi}{4\pi}F^{\mathrm{s}}e^{-\tau/\mu_0}}_{solar\ pseudo-source} \tag{59}$$

where I_{d}^{+} and I_{d}^{-} are the diffuse upward and downward radiances.

Prototype Problem 3: Beam Incidence (3)

As before, we take the sum and difference of these equations:

$$\bar{\mu}\frac{d(I_{\rm d}^{+} - I_{\rm d}^{-})}{d\tau} = (1-\varpi)(I_{\rm d}^{+} + I_{\rm d}^{-}) - \frac{\varpi}{2\pi}F^{\rm s}e^{-\tau/\mu_0} \quad (60)$$

$$\bar{\mu}\frac{d(I_{\rm d}^{+} + I_{\rm d}^{-})}{d\tau} = (I_{\rm d}^{+} - I_{\rm d}^{-}). \quad (61)$$

- Differentiating Eq. 61 and substituting into Eq. 60, we find:

$$\boxed{\frac{d^2(I_{\rm d}^{+} + I_{\rm d}^{-})}{d\tau^2} = \frac{(1-\varpi)}{\bar{\mu}^2}(I_{\rm d}^{+} + I_{\rm d}^{-}) - \frac{\varpi}{2\pi}F^{\rm s}e^{-\tau/\mu_0}.}$$

- Similarly, by differentiating Eq. 60 and substituting into Eq. 61, we get:

$$\boxed{\frac{d^2(I_{\rm d}^{+} - I_{\rm d}^{-})}{d\tau^2} = \frac{(1-\varpi)}{\bar{\mu}^2}(I_{\rm d}^{+} - I_{\rm d}^{-}) + \frac{\varpi\bar{\mu}}{2\pi\mu_0}F^{\rm s}e^{-\tau/\mu_0}.}$$

Prototype Problem 3: Beam Incidence (4)

- We may use the same solution method used earlier for *Prototype Problem 2*. Thus, the homogeneous solution can be written:

$$I_{\rm d}^{+} = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau}; \qquad I_{\rm d}^{-} = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}$$

where Γ and ρ_∞ are defined in Eqs. 19 and 22.

- We guess that the particular solution is proportional to $e^{-\tau/\mu_0}$. Thus, the complete solutions become:

$$I_{\rm d}^{+} = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau} + Z^{+}e^{-\tau/\mu_0}$$
$$I_{\rm d}^{-} = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau} + Z^{-}e^{-\tau/\mu_0}$$

where Z^{+} and Z^{-} are constants to be determined.

- Substituting into Eqs. 58 and 59, we find:

$$Z^{+} + Z^{-} = -\frac{\varpi F^s \mu_0^2}{2\pi\bar{\mu}^2(1-\Gamma^2\mu_0^2)}; \quad Z^{+} - Z^{-} = \frac{\varpi F^s \mu_0\bar{\mu}}{2\pi\bar{\mu}^2(1-\Gamma^2\mu_0^2)}. \tag{62}$$

Prototype Problem 3: Beam Incidence (5)

- Solving for Z^+ and Z^-, we have:

$$Z^+ = \frac{\varpi F^{\rm s}\mu_0(\bar{\mu} - \mu_0)}{4\pi\bar{\mu}^2(1 - \Gamma^2\mu_0^2)}; \qquad Z^- = -\frac{\varpi F^{\rm s}\mu_0(\mu_0 + \bar{\mu})}{4\pi\bar{\mu}^2(1 - \Gamma^2\mu_0^2)}. \tag{63}$$

- We apply boundary conditions for the diffuse radiance $I_{\rm d}^-(\tau = 0) = 0$ and $I_{\rm d}^+(\tau^*) = 0$ to obtain two equations for A and D. After some manipulation we find:

$$A = -\frac{\varpi F^{\rm s}\mu_0}{4\pi\bar{\mu}^2(1 - \Gamma^2\mu_0^2)\mathcal{D}}\left[\rho_\infty(\bar{\mu} + \mu_0)e^{-\Gamma\tau^*} + (\bar{\mu} - \mu_0)e^{-\tau^*/\mu_0}\right] \tag{64}$$

$$D = \frac{\varpi F^{\rm s}\mu_0}{4\pi\bar{\mu}^2(1 - \Gamma^2\mu_0^2)\mathcal{D}}\left[(\bar{\mu} + \mu_0)e^{\Gamma\tau^*} + \rho_\infty(\bar{\mu} - \mu_0)e^{-\tau^*/\mu_0}\right] \tag{65}$$

with as defined previously (Eq. 27):

$$\mathcal{D} \equiv e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}.$$

Prototype Problem 3: Beam Incidence (6)

- We may now solve for the source function, irradiance etc. For example, the source function is:

$$S(\tau) = \frac{\varpi}{2}(I_{\rm d}^{+} + I_{\rm d}^{-}) + \frac{\varpi F^{\rm s}}{4\pi} e^{-\tau/\mu_0}. \tag{66}$$

- For a semi-infinite medium ($\tau^* \to \infty$), the condition that the solution be bounded [$S(\tau)e^{\tau} \to 0$], shows that the positive exponentials must be discarded, so that $A = 0$. The constant D reduces to ($\mathcal{D} \to 1$):

$$D = \frac{\varpi F^{\rm s} \mu_0 (\bar{\mu} + \mu_0)}{4\pi \bar{\mu}^2 (1 - \Gamma^2 \mu_0^2)}.$$

Prototype Problem 3: Beam Incidence (7)

- The diffuse radiances become:

$$
\begin{aligned}
I_{\rm d}^{+}(\tau) &= \rho_\infty D e^{-\Gamma\tau} + Z^{+} e^{-\tau/\mu_0} \\
&= \frac{\varpi F^{\rm s}\mu_0}{4\pi\bar{\mu}^2(1-\Gamma^2\mu_0^2)}[\rho_\infty(\bar{\mu}+\mu_0)e^{-\Gamma\tau} + (\bar{\mu}-\mu_0)e^{-\tau/\bar{\mu}}] \qquad (67)\\
I_{\rm d}^{-}(\tau) &= D e^{-\Gamma\tau} + Z^{-} e^{-\Gamma/\mu_0} \\
&= \frac{\varpi F^{\rm s}\mu_0}{4\pi\bar{\mu}^2(1-\Gamma^2\mu_0^2)}[(\bar{\mu}+\mu_0)e^{-\Gamma\tau} - (\bar{\mu}+\mu_0)e^{-\tau/\mu_0}] \qquad (68)
\end{aligned}
$$

and the source function becomes (Eq. 66):

$$
\begin{aligned}
S(\tau) &= \frac{\varpi F^{\rm s}}{4\pi}\{\frac{\varpi\mu_0}{\bar{\mu}^2(1-\Gamma^2\mu_0^2)}[\frac{1}{2}(\bar{\mu}+\mu_0)(1+\rho_\infty)e^{-\Gamma\tau} \\
&\quad - \mu_0 e^{-\tau/\mu_0}] + e^{-\tau/\mu_0}\}.
\end{aligned}
\qquad (69)
$$

Prototype Problem 3: Beam Incidence (8)

- We may ask: what happens if the denominator $(1-\Gamma^2\mu_0^2)$ is zero in the equations for $I_{\rm d}^{\pm}$? This problem can occur if the Sun is at a specific location in the sky.
- This problem is a so-called *removable singularity*, that can be "cured" by the application of *L'Hôpital's rule*, which leads to a new algebraic form that varies as $\tau\exp(-\tau/\mu_0)$.
- In computational work it is usually sufficient to use numerical "dithering" by which μ_0 is changed slightly away from the "singular value."
- This artifice produces satisfactory results, and avoids the "inconvenience" of having to deal with a special case involving a different solution.

The total net irradiance and heating rate become:

$$
\begin{aligned}
F(\tau) &= 2\pi\bar{\mu}(I_{\rm d}^{+} - I_{\rm d}^{-}) - \mu_0 F^{\rm s} e^{-\tau/\mu_0} \\
&= \frac{\varpi F^{\rm s}\mu_0(\bar{\mu}+\mu_0)}{2\bar{\mu}(1-\Gamma^2\mu_0^2)}[\rho_\infty(\bar{\mu}+\mu_0)e^{-\Gamma\tau} - 2\mu_0 e^{-\tau/\mu_0}] - \mu_0 F^{\rm s} e^{-\tau/\mu_0}
\end{aligned}
\tag{70}
$$

and

Prototype Problem 3: Beam Incidence (9)

$$\begin{aligned}\mathcal{H}(\tau) &= 2\pi\alpha(I_{\rm d}^{+} + I_{\rm d}^{-}) + \alpha F^{\rm s} e^{-\tau/\mu_0} \\ &= 2\pi\alpha \frac{\varpi F^{\rm s}\mu_0(\bar{\mu}+\mu_0)}{4\pi\bar{\mu}^2(1-\Gamma^2\mu_0^2)}[(1+\rho_\infty)(\bar{\mu}+\mu_0)e^{-\Gamma\tau} \\ &\quad - 2\mu_0 e^{-\tau/\mu_0}] + \alpha F^{\rm s} e^{-\tau/\mu_0}. \end{aligned} \tag{71}$$

- Note that we have added the terms $-\mu_0 F^{\rm s} e^{-\tau/\mu_0}$ in the irradiance equation and the term $\alpha F^{\rm s} e^{-\tau/\mu_0}$ in the heating equation to include the contributions from the solar component.

- For a discussion of the accuracy of the two-stream approximation, see §S.4.5 of Appendix S.

Conservative Scattering in a Finite Slab

It is instructive to solve the radiative transfer problem anew in this case, since a new feature occurs, namely that the homogeneous solution for $I^{+} + I^{-}$ is now a linear function of τ, (say, $B\tau + C$). We will not show the details of the solution (see Exercise 7.4). The results are given below for Prototype Problem 3:

Prototype Problem 3: Beam Incidence (10)

$$I_{\rm d}^{+}(\tau) = \frac{F^{\rm s}m}{4\pi}\Big[\frac{(m+1)(\tau^* - \tau) + (m-1)(\tau + 2\bar{\mu})e^{-\tau^*/\mu_0}}{(\tau^* + 2\bar{\mu})} - (m-1)e^{-\tau/\mu_0}\Big] \quad (72)$$

$$I_{\rm d}^{-}(\tau) = \frac{F^{\rm s}m}{4\pi}\Big[\frac{(m+1)(\tau^* - \tau + 2\bar{\mu}) + (m-1)\tau e^{-\tau^*/\mu_0}}{(\tau^* + 2\bar{\mu})} - (m+1)e^{-\tau/\mu_0}\Big] \quad (73)$$

$$S(\tau) = \frac{F^{\rm s}m}{4\pi(\tau^* + 2\bar{\mu})} \quad [(m+1)(\tau^* - \tau + \bar{\mu}) + (m-1)(\tau + \bar{\mu})e^{-\tau^*/\mu_0}$$

$$- \; m(\tau^* + 2\bar{\mu})e^{-\tau/\mu_0}] + \frac{F^{\rm s}}{4\pi}e^{-\tau/\mu_0} \quad (74)$$

$$F(\tau) = -\frac{F^{\rm s}\mu_0\bar{\mu}}{(\tau^* + 2\bar{\mu})}[(1+m) + (1-m)e^{-\tau^*/\mu_0}] \quad (75)$$

$$\rho(-\mu_0, 2\pi) = \frac{F^{+}(0)}{\mu_0 F^{\rm s}} = \frac{\tau^* + (\bar{\mu} - \mu_0)(1 - e^{-\tau^*/\mu_0})}{\tau^* + 2\bar{\mu}} \quad (76)$$

$$\mathcal{T}(-\mu_0, -2\pi) = \frac{F^{-}(\tau^*)}{\mu_0 F^{\rm s}} = \frac{\bar{\mu} + \mu_0 + (\bar{\mu} - \mu_0)e^{-\tau^*/\mu_0}}{\tau^* + 2\bar{\mu}}, \quad (77)$$

where $m \equiv \mu_0/\bar{\mu}$. Since Eq. 73 is the diffuse radiance, the total transmittance given by Eq. 77 is $\mathcal{T}(-\mu_0, -2\pi) = \mathcal{T}_{\rm d}(-\mu_0, -2\pi) + e^{-\tau^*/\mu_0} = (2\pi\bar{\mu}I_{\rm d}^{-}(\tau^*)/\mu_0 F^{\rm s}) + e^{-\tau^*/\mu_0}$. The sum $\rho(-\mu_0, 2\pi) + \mathcal{T}(-\mu_0, -2\pi) = 1$ can be easily verified.

Prototype Problem 3: Beam Incidence (11)

Consider the limit of large optical depth.

For $\tau^* \to \infty$, $F(\tau^*) \to 0$, $\rho(-\mu_0, 2\pi) \to 1$, $\mathcal{T}(-\mu_0, 2\pi) \to 0$, and the source function becomes

$$S(\tau) = \frac{F^{\mathrm{s}}}{4\pi}[(1 - m^2)e^{-\tau/\mu_0} + m(m + 1)]. \tag{78}$$

From this expression, we see that our two-stream approximation yields $S(\tau \to \infty)/S(0) = m = \sqrt{3}\mu_0$ when $\bar{\mu} = 1/\sqrt{3}$, a result that is *exact*.†

The angular distribution of the radiance for Prototype Problem 3 can be obtained in the same fashion as in Prototype Problem 1, but the algebra is rather daunting. As shown in Appendix S (§S.4.3) a "short-cut" is possible, provided we are interested only in the emergent radiances.

† See Chandrasekhar, 1960, Eq. 131, p. 87.

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (1)

- Two-stream approximations are used primarily to compute irradiances and mean radiances in a slab geometry.

- Since the irradiance and mean radiance depend only on the azimuthally-averaged radiation field, we are interested in simple solutions to the azimuthally-averaged RTE valid for anisotropic scattering:

$$u\frac{dI_{\rm d}(\tau,u)}{d\tau} = I_{\rm d}(\tau,u) - \frac{\varpi}{2}\int_{-1}^{1} du' p(u',u) I_{\rm d}(\tau,u') - S^*(\tau,u) \qquad (79)$$

  where we have ignored thermal emission.

- To obtain approximate solutions, we proceed by integrating Eq. 79 over each hemisphere to find two coupled, first-order differential equations for hemispherically-averaged upward and downward radiance "streams."

- This approach leads to the usual **two-stream approximation (TSA).** We can obtain a similar result by replacing the integral in Eq. 79 by a two-term numerical quadrature (see Example 7.10).

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (2)

We may alternatively proceed by approximating the angular dependence by a linear polynomial, $I(\tau, u) = I_0(\tau) + uI_1(\tau)$, and taking angular moments of Eq. 79, to arrive at two coupled equations for I_0 and I_1 (see Appendix S, §S.4.4, for details)

$$\frac{dI_1}{d\tau} = \frac{1}{\langle u\rangle_2}(1-\varpi)I_0 - \frac{\varpi F^{\mathrm{s}}}{4\pi\langle u\rangle_2}e^{-\tau/\mu_0} \tag{80}$$

$$\frac{dI_0}{d\tau} = (1 - 3g\varpi\langle u\rangle_2)I_1 + \frac{3\varpi F^{\mathrm{s}}}{4\pi}g\mu_0 e^{-\tau/\mu_0}, \tag{81}$$

where‡

$$\langle u\rangle_2 \equiv \frac{1}{2}\int_{-1}^{1} du u^2.$$

This approach is usually referred to as the *Eddington approximation.*

In the following, we examine both the Eddington and the two-stream approximation.

- We shall be particularly interested in exposing the similarities and differences between these two approaches.

‡Note that the symbol $\langle u\rangle_2$ is used instead of the numerical value 1/3 to facilitate comparison with the two-stream approximation at the end of this section.

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (3)

- We expand the scattering phase function in Legendre polynomials $P_\ell(u)$, and find the azimuthally-averaged phase function to be:

$$p(u', u) = \sum_{\ell=0}^{\infty} (2l + 1)\chi_\ell P_\ell(u) P_\ell(u') \quad \longleftarrow \quad \textbf{Legendre polynomial expansion}$$

where the moments or expansion coefficients are given by:

$$\chi_\ell = \frac{1}{2P_\ell(u)} \int_{-1}^{+1} du' p(u', u) P_\ell(u').$$

- In the TSA, we normally retain only two terms: (1) the zeroth moment which is unity because of the normalization ($\chi_0 = 1$), and (2) the first moment which we refer to as the **asymmetry factor**: $g \equiv \chi_1$. Then:

since $P_0(u) = 1$ and $P_1(u) = u$, we obtain the two-term approximation (TTA):

$$p(u', u) \approx \sum_{\ell=0}^{1} (2\ell + 1)\chi_\ell P_\ell(u) P_\ell(u') = 1 + 3gu'u \quad \longleftarrow \quad \textbf{TTA}.$$

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (4)

We start by writing Eq. 79 in terms of the half-range radiances:

$$\mu \frac{dI_{\rm d}^{+}(\tau,\mu)}{d\tau} = I_{\rm d}^{+}(\tau,\mu) - S^{+}(\tau,\mu) \tag{82}$$

$$-\mu \frac{dI_{\rm d}^{-}(\tau,\mu)}{d\tau} = I_{\rm d}^{-}(\tau,\mu) - S^{-}(\tau,\mu) \tag{83}$$

$$S^{\pm}(\tau,\mu) \equiv -\frac{\varpi}{2}\int_0^1 d\mu' p(-\mu',\pm\mu) I_{\rm d}^{-}(\tau,\mu') - \frac{\varpi}{2}\int_0^1 d\mu' p(\mu',\pm\mu) I_{\rm d}^{+}(\tau,\mu') - \frac{\varpi F^{\rm s}}{4\pi} p(-\mu_0,\pm\mu) e^{-\tau/\mu_0}. \tag{84}$$

These two equations are exact for the slab problem. We proceed by integrating each equation of these equations over the appropriate hemisphere by applying the operator $\int_0^1 d\mu \cdots$:

- If $I^{+}(\tau,\mu)$ and $I^{-}(\tau,\mu)$ are replaced by their averages $I^{+}(\tau)$ and $I^{-}(\tau)$ over the upper and lower hemisphere, respectively, and μ^{+} and μ^{-} are replaced by the same average value $\bar{\mu}$, this approach leads to the following pair of coupled equations for $I^{\pm}$ (dropping the 'd' subscript):

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (5)

Two-Stream Approximation:

$$\bar{\mu}\frac{dI^+}{d\tau} = I^+ - \varpi(1-b)I^+ - \varpi b I^- - S^{*+} \tag{85}$$

$$-\bar{\mu}\frac{dI^-}{d\tau} = I^- - \varpi(1-b)I^- - \varpi b I^+ - S^{*-} \tag{86}$$

where

$$S^{*+} \equiv \frac{\varpi F^{\rm s}}{2\pi} b(\mu_0) e^{-\tau/\mu_0} \equiv X^+(\tau) e^{-\tau/\mu_0}$$

$$S^{*-} \equiv \frac{\varpi F^{\rm s}}{2\pi} [1 - b(\mu_0)] e^{-\tau/\mu_0} \equiv X^-(\tau) e^{-\tau/\mu_0} \tag{87}$$

$$X^+(\tau) \equiv \frac{\varpi}{2\pi} F^{\rm s} b(\mu_0); \qquad X^-(\tau) \equiv \frac{\varpi}{2\pi} F^{\rm s}[1 - b(\mu_0)] \tag{88}$$

and b and $b(\mu_0)$ are backscattering ratios (or fractions) to be defined below.

- Equations 85 and 86 are valid for anisotropic scattering. For isotropic scattering, ($p = 1$ and $b = \frac{1}{2}$, $b(\mu) = \frac{1}{2}$), they reduce to those derived previously.

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (6)

We have derived two sets of differential equations (Eqs. 80 and 81 and Eqs. 85 and 86) from similar assumptions. What is the relationship, if any, between them? To answer this question, we will bring Eqs. 85 and 86 into a form similar to Eqs. 80 and 81 by using the change of variable

$$I^{\pm}(\tau) = I_0 \pm \bar{\mu} I_1$$

consistent with the Eddington approximation. By first adding Eqs. 85 and 86, and then subtracting 85 from 86, we find after some manipulation that Eqs. 85 and 86 are equivalent to (Appendix S)

$$\frac{dI_1}{d\tau} = \frac{1-\varpi}{\bar{\mu}^2} I_0 - \frac{\varpi}{4\pi\bar{\mu}^2} F^{\mathrm{s}} e^{-\tau/\mu_0} \tag{89}$$

$$\frac{dI_0}{d\tau} = (1 - 3g\varpi\bar{\mu}^2) I_1 + \frac{3\varpi}{4\pi} g\mu_0 F^{\mathrm{s}} e^{-\tau/\mu_0}. \tag{90}$$

By comparing Eqs. 80 and 81 and 89 and 90, we conclude that *the equations describing the Eddington and two-stream approximations are identical provided* $\langle u \rangle_2 = \bar{\mu}^2$.

Anisotropic Scattering: Two-Stream Approximation – Beam Incidence (7)

Thus, since $\langle u \rangle_2 = \frac{1}{3}$, the choice $\bar{\mu} = 1/\sqrt{3}$ makes the governing equations for the two methods the same. Therefore:

- any remaining difference between the two must stem from different boundary conditions, which is readily seen as follows:
- A homogeneous boundary condition for the downward diffuse radiance consistent with the two-stream approximation leads to the boundary condition

$$I^{-}(0) = I_0 - \bar{\mu} I_1 = 0.$$

If, on the other hand, we require the downward diffuse *irradiance* to be zero at the upper boundary (common practice in the Eddington approximation), then we find

$$I_0 - \frac{2}{3} I_1 = 0.$$

As we shall see later (Chapter 9), the value $\bar{\mu} = 1/\sqrt{3}$ for the average cosine follows from applying full-range Gaussian quadrature while a half-range Gaussian quadrature would lead to $\bar{\mu} = \frac{1}{2}$.

Anisotropic Scattering – Beam Incidence: The Backscattering Coefficients (1)

- The **backscattering ratios** are defined as:

$$b(\mu) \equiv \frac{1}{2}\int_0^1 d\mu'\, p(-\mu',\mu) = \frac{1}{2}\int_0^1 d\mu'\, p(\mu',-\mu) \tag{91}$$

$$b \equiv \int_0^1 d\mu\, b(\mu) = \frac{1}{2}\int_0^1 d\mu \int_0^1 d\mu'\, p(-\mu',\mu) = \frac{1}{2}\int_0^1 d\mu \int_0^1 d\mu'\, p(\mu',-\mu) \tag{92}$$

$$1 - b = \frac{1}{2}\int_0^1 d\mu \int_0^1 d\mu'\, p(\mu',\mu) = \frac{1}{2}\int_0^1 d\mu \int_0^1 d\mu'\, p(-\mu',-\mu). \tag{93}$$

- We have used the **Reciprocity Relations** satisfied by the scattering phase function, $p(-\mu',\mu) = p(\mu',-\mu)$; $p(-\mu',-\mu) = p(\mu',\mu)$, as well as the normalization property.

- The backscattering ratios $b(\mu)$ and b (see Eqs. 91 and 92) define the fraction of the energy that is scattered into the backward hemisphere. Of course, $1 - b$ or $1 - b(\mu_0)$ is the fraction that is forward-scattered.

- If we use the **TTA**: $p(u',u) \approx 1 + 3gu'u$ and **quadrature** choosing $\bar{\mu} = 1/\sqrt{3}$, then $b = \frac{1}{2}\int_0^1 d\mu \int_0^1 \mu'(1 - 3g\mu\mu') \approx \frac{1}{2}(1 - 3g\bar{\mu}^2) = \frac{1}{2}(1-g)$: the backscattering ratio is related to the asymmetry factor through $g = (1 - 2b)$.

Anisotropic Scattering – Beam Incidence: The Backscattering Coefficients (2)

We normally do not use the phase function itself, but rather its expansion:

$$p(u', u) = \sum_{\ell=0}^{2N-1} (2\ell+1)\chi_\ell P_\ell(u') P_\ell(u) \quad \longleftarrow \quad \textbf{Legendre polynomial expansion}$$

Substituting this expansion into Eqs. 91 and 92, we find:

$$\begin{aligned} b(\mu) &= \frac{1}{2}\int_0^1 d\mu' p(-\mu', \mu) = \frac{1}{2}\sum_{\ell=0}^{2N-1} (-1)^\ell (2\ell+1)\chi_\ell P_\ell(\mu) \int_0^1 d\mu' P_\ell(\mu') \\ &\equiv \sum_{\ell=0}^{2N-1} b_\ell(\mu) \end{aligned} \tag{94}$$

where we have used the relation $P_\ell(-\mu) = (-1)^\ell P_\ell(\mu)$ satisfied by the Legendre polynomials, and defined $b_\ell(\mu) \equiv \frac{1}{2}(-1)^\ell (2\ell+1)\chi_\ell P_\ell(\mu) \int_0^1 d\mu' P_\ell(\mu')$.

- Using Eqs. 92 [$b = \int_0^1 d\mu b(\mu)$] and 94, we obtain:

$$b = \int_0^1 d\mu b(\mu) = \frac{1}{2}\sum_{\ell=0}^{2N-1} (-1)^\ell (2\ell+1)\chi_\ell [\int_0^1 d\mu P_\ell(\mu)]^2. \tag{95}$$

- For $N = 1$ these formulas yield $b(\mu) = \frac{1}{2}(1 - \frac{3}{2}g\mu)$ and $b = \frac{1}{2}(1 - \frac{3}{4}g)$, which are identical with the results obtained using quadrature with the choice $\bar{\mu} = 1/2$.

Anisotropic Scattering – Beam Incidence: The Backscattering Coefficients (3)

- Note that whereas use of the TTA and quadrature $\bar{\mu} = 1/\sqrt{3}$:

$$p(u', u) \approx 1 + 3gu'u \;\Rightarrow\; b(\mu) \approx \frac{1}{2}(1 - 3g\bar{\mu}'\mu) \;\Rightarrow\; b \approx \frac{1}{2}(1 - g)$$

  yields $b = 0$ for complete forward scattering ($g = 1$) and $b = 1$ for complete backscattering ($g = -1$), we obtain the 'unphysical' results $b = 1/8$ and $b = 7/8$, respectively for $\bar{\mu} = 1/2$ from $b = \frac{1}{2}(1 - \frac{3}{4}g)$. Thus, in the TSA:

- we may want to adopt $\bar{\mu} = 1/\sqrt{3}$ and use $b(\mu) = \frac{1}{2}(1 - \frac{3}{\sqrt{3}}g\mu)$ and $b = \frac{1}{2}(1 - g)$ to compute the backscattering coefficients from the asymmetry factor, g.

- We can determine these coefficients more accurately from Eqs. 91 and 92 or the "summation" formulas above by numerical evaluation of the double integrals.

- Finally, as discussed in Chapter 6, the backscattering ratio can be computed exactly using Eq. 6.23, involving only a single integral:

$$b = \frac{1}{2}\int_{\pi/2}^{\pi} d\Theta \sin\Theta p(\tau, \cos\Theta) = \frac{1}{2}\int_0^1 dy p(\tau, -y).$$

Anisotropic Scattering – Beam Incidence: How Do We Solve the Two-Stream Equations?

Recall:

Two-Stream Equations:

$$\bar{\mu}\frac{dI_{\rm d}^+}{d\tau} = I_{\rm d}^+ - \varpi(1-b)I_{\rm d}^+ - \varpi b I_{\rm d}^- - S^{*+} \tag{96}$$

$$-\bar{\mu}\frac{dI_{\rm d}^-}{d\tau} = I_{\rm d}^- - \varpi(1-b)I_{\rm d}^- - \varpi b I_{\rm d}^+ - S^{*-} \tag{97}$$

where:

$$S^{*+} \equiv \frac{\varpi F^{\rm s}}{2\pi}b(\mu_0)e^{-\tau/\mu_0} \equiv X^+(\tau)e^{-\tau/\mu_0} \tag{98}$$

$$S^{*-} \equiv \frac{\varpi F^{\rm s}}{2\pi}[1-b(\mu_0)]e^{-\tau/\mu_0} \equiv X^-(\tau)e^{-\tau/\mu_0} \tag{99}$$

$$X^+(\tau) \equiv \frac{\varpi}{2\pi}F^{\rm s}b(\mu_0); \qquad X^-(\tau) \equiv \frac{\varpi}{2\pi}F^{\rm s}[1-b(\mu_0)]. \tag{100}$$

Two-Stream Solutions for Anisotropic Scattering – Beam Incidence (1)

Focusing first on the homogeneous solution, we add and subtract Eqs. 96 and 97 to obtain:

$$\frac{d(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-})}{d\tau} = -(\alpha - \beta)(I_{\mathrm{d}}^{+} - I_{\mathrm{d}}^{-}) \tag{101}$$

$$\frac{d(I_{\mathrm{d}}^{+} - I_{\mathrm{d}}^{-})}{d\tau} = -(\alpha + \beta)(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-}) \tag{102}$$

where we have defined $\alpha \equiv -[1 - \varpi(1-b)]/\bar{\mu}$ and $\beta \equiv \varpi b/\bar{\mu}$.

- By differentiating one equation and substituting into the second, we obtain the following uncoupled equations to solve:

$$\frac{d^2(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-})}{d\tau^2} = \Gamma^2(I_{\mathrm{d}}^{+} + I_{\mathrm{d}}^{-}); \qquad \frac{d^2(I_{\mathrm{d}}^{+} - I_{\mathrm{d}}^{-})}{d\tau^2} = \Gamma^2(I_{\mathrm{d}}^{+} - I_{\mathrm{d}}^{-}) \tag{103}$$

where

$$\Gamma = \sqrt{(\alpha - \beta)(\alpha + \beta)} = (1/\bar{\mu})\sqrt{(1-\varpi)(1-\varpi+2\varpi b)}. \tag{104}$$

Hence in the limit of isotropic scattering (see Eqs. 19) as $b \to 1/2$:

$$\Gamma \to \sqrt{(1-\varpi)}/\bar{\mu}.$$

Two-Stream Solutions for Anisotropic Scattering – Beam Incidence (2)

As in the case of isotropic scattering, the homogeneous solutions are:

$$I_{\rm d}^{+}(\tau) = Ae^{\Gamma\tau} + Be^{-\Gamma\tau} = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau} \tag{105}$$

$$I_{\rm d}^{-}(\tau) = Ce^{\Gamma\tau} + De^{-\Gamma\tau} = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}. \tag{106}$$

- The coefficients A, B, C, and D are NOT all independent as pointed out previously. The relation between them is found by substituting Eqs. 105 and 106 into Eqs. 96 and 97, yielding:

$$\frac{C}{A} = \frac{B}{D} = \frac{\sqrt{1-\varpi+2\varpi b} - \sqrt{1-\varpi}}{\sqrt{1-\varpi+2\varpi b} + \sqrt{1-\varpi}} \equiv \rho_\infty.$$

Particular Solution:

Equations 96–100 ($S^{*\pm} \propto e^{-\tau/\mu_0}$) suggest seeking a particular solution of the form:

$$I_{\rm d}^{\pm} = Z^{\pm}e^{-\tau/\mu_0}. \tag{107}$$

Two-Stream Solutions for Anisotropic Scattering – Beam Incidence (3)

Substitution of Eq. 107 into Eqs. 96 and 97 yields:

$$Z^{\pm} = \frac{\varpi b X^{\mp} + [1 - \varpi + \varpi b \mp \bar{\mu}/\mu_0] X^{\pm}}{(1-\varpi)(1-\varpi+2\varpi b) - (\bar{\mu}/\mu_0)^2}$$

where X^+ and $X^-$are: $X^+(\tau) \equiv \frac{\varpi}{2\pi} F^{\mathrm{s}} b(\mu_0)$; $\quad X^-(\tau) \equiv \frac{\varpi}{2\pi} F^{\mathrm{s}}[1 - b(\mu_0)]$.

- Note that if we set $b = \frac{1}{2}$ $(g = 0)$ and observe that in this case $X^+ = X^- = \frac{\varpi F^{\mathrm{s}}}{4\pi}$, it can be verified that $\Gamma \to \sqrt{(1-\varpi)}/\bar{\mu}$ (Eqs. 19) and $Z^{\pm} \to Z^+ = \frac{\varpi F^{\mathrm{s}} \mu_0 (\bar{\mu} - \mu_0)}{4\pi \bar{\mu}^2 (1 - \Gamma^2 \mu_0^2)}$; $\quad Z^- = -\frac{\varpi F^{\mathrm{s}} \mu_0 (\mu_0 + \bar{\mu})}{4\pi \bar{\mu}^2 (1 - \Gamma^2 \mu_0^2)}$ (Eqs. 63).
- It is also clear that for $b = \frac{1}{2}$ we recover the earlier result for ρ_∞ (see Eq. 22).
- We determine the constants A and D in Eqs. 105 and 106 from the homogeneous radiation boundary conditions appropriate for the diffuse radiances:

$$A = \frac{(-Z^+ e^{-\tau^*/\mu_0} + Z^- \rho_\infty e^{-\Gamma\tau^*})}{\mathcal{D}}; \qquad D = \frac{(Z^+ \rho_\infty e^{-\tau^*/\mu_0} - Z^- e^{\Gamma\tau^*})}{\mathcal{D}}$$

where as defined previously (Eq. 27): $\mathcal{D} \equiv e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}$.

Two-Stream Solutions for Anisotropic Scattering – Beam Incidence (4)

- These solutions satisfy the differential Eqs. 96 and 97, and also obey homogeneous boundary conditions.
- It is easy to show that in the limit of **isotropic scattering** the expressions for A and D above reduce to those in Eqs. 64 and 65, as expected. The solutions for the diffuse radiances are:

$$
\begin{aligned}
I_{\mathrm{d}}^{+}(\tau) &= \frac{1}{\mathcal{D}}[(-Z^{+}e^{-\tau/\mu_0} + Z^{-}\rho_{\infty}e^{-\Gamma\tau^*})e^{\Gamma\tau} \\
&\quad + \rho_{\infty}(Z^{+}\rho_{\infty}e^{-\tau^*/\mu_0} - Z^{-}e^{\Gamma\tau^*})e^{-\Gamma\tau}] + Z^{+}e^{-\tau/\mu_o}
\end{aligned}
$$

$$
\begin{aligned}
I_{\mathrm{d}}^{-}(\tau) &= \frac{1}{\mathcal{D}}[(-Z^{+}e^{-\tau/\mu_0} + Z^{-}\rho_{\infty}e^{-\Gamma\tau^*})\rho_{\infty}e^{\Gamma\tau} \\
&\quad + (Z^{+}\rho_{\infty}e^{-\tau^*/\mu_0} - Z^{-}e^{\Gamma\tau^*})e^{-\Gamma\tau}] + Z^{-}e^{-\tau/\mu_0}.
\end{aligned}
$$

Two-Stream Solutions for Anisotropic Scattering – Beam Incidence (5)

- We can now solve for the half-range source functions, the irradiance, and the heating rate:

$$S^+(\tau) = \varpi(1-b)I_{\rm d}^+(\tau) + \varpi b I_{\rm d}^-(\tau) + \frac{\varpi F^{\rm s} e^{-\tau/\mu_0}}{2\pi} b(\mu_0) \tag{108}$$

$$S^-(\tau) = \varpi(1-b)I_{\rm d}^-(\tau) + \varpi b I_{\rm d}^+(\tau) + \frac{\varpi F^{\rm s} e^{-\tau/\mu_0}}{2\pi} [1 - b(\mu_0)] \tag{109}$$

$$F(\tau) = 2\pi\bar{\mu}[I_{\rm d}^+(\tau) - I_{\rm d}^-(\tau)] - \mu_0 F^{\rm s} e^{-\tau/\mu_0} \tag{110}$$

$$\mathcal{H}(\tau) = 2\pi\alpha[I_{\rm d}^+(\tau) + I_{\rm d}^-(\tau)] + \alpha F^{\rm s} e^{-\tau/\mu_0}.$$

.

Scaling Approximations for Anisotropic Scattering (1)

- In §6.5 we noted that an accurate representation of a sharply-peaked phase function typically requires several hundred terms in a Legendre polynomial expansion.

- By making the approximation that photons scattered within this peak are not scattered at all, we found the RTE to become more tractable (see Eq. 6.48):

$$\hat{p}_{\delta-\mathrm{M}}(\cos\Theta) \equiv 2f\delta(1-\cos\Theta) + (1-f)\sum_{\ell=0}^{M-1}(2\ell+1)\hat{\chi}_\ell P_\ell(\cos\Theta).$$

- This artifice is known as a **scaling approximation** (see §6.6), and takes on various forms depending upon the choice of the truncation.

- We found that in the δ-isotropic approximation: $\hat{p}_{\delta-\mathrm{M}}(\cos\Theta) \equiv 2f\delta(1-\cos\Theta) + (1-f)$ the scaled RTE corresponds to an isotropic scattering problem:

$$\mu\frac{dI^{\pm}(\hat{\tau},\mu)}{d\tau} = I^{\pm}(\hat{\tau},\mu) - \frac{\hat{\varpi}}{2}\int_0^1 d\mu'[I^+(\hat{\tau},\mu') + I^-(\hat{\tau},\mu')]$$

but with a scaled optical depth $d\hat{\tau} = (1-\varpi f)d\tau$ and a scaled single-scattering albedo $\hat{\varpi} = (1-f)\varpi/(1-\varpi f)$. The value of f is somewhat arbitrary, but a good choice is $f = g$, where g is the asymmetry factor.

Scaling Approximations for Anisotropic Scattering (2)

- Since we have solved the RTE above in the two-stream approximation for three prototype problems, it is a trivial matter to rewrite the solutions in terms of the scaled parameters, $\hat{\varpi}$ and $\hat{\tau}$.
- We will write the asymmetry factor in terms of the backscattering coefficient, $b = (1-g)/2$. We use as an example the conservative scattering limit (no absorption), $\hat{\varpi} = 1$ and $\hat{\tau} = (1-g)\tau = 2b\tau$.
- For **Prototype Problem 3** the scaled solutions for the reflectance and transmittance are taken from Eqs. 76 and 77 (setting $\hat{\tau}^* = (1-g)\tau^* = 2b\tau^*$):

$$\rho(-\mu_0, 2\pi) = \frac{2b\tau^* + (\bar{\mu} - \mu_0)(1 - e^{-2b\tau^*/\mu_0})}{2b\tau^* + 2\bar{\mu}} \tag{111}$$

$$\mathcal{T}(\mu_0, 2\pi) = \frac{\bar{\mu} + \mu_0 + (\bar{\mu} - \mu_0)e^{-2b\tau^*/\mu_0}}{2b\tau^* + 2\bar{\mu}}. \tag{112}$$