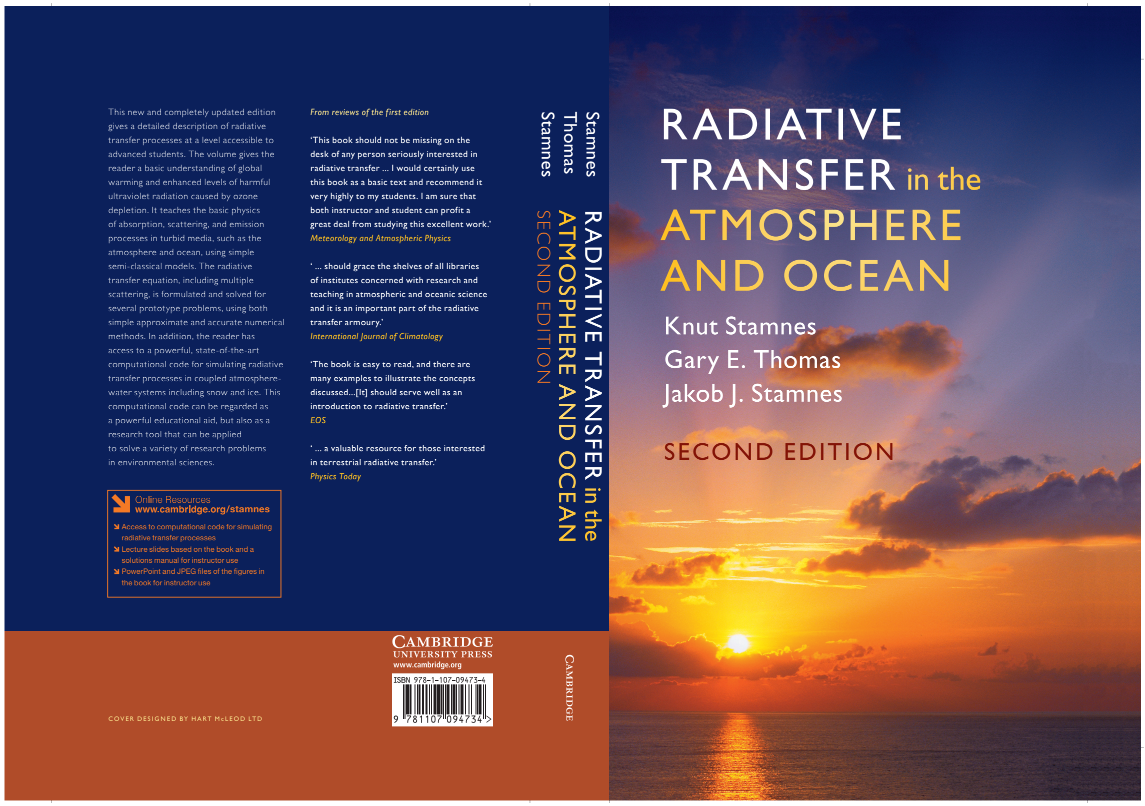
Lecture *Notes:*
State Variables



Based on Chapter 2 in K. Stamnes, G. E. Thomas, and J. J. Stamnes, Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, 2017.

Basic State Variables – Scalar versus vector radiance (1)

In many instances we are mostly concerned with the flow of radiative *energy* through atmospheres and oceans. Then:

- We can ignore polarization effects: *Disregard* the Q, U, and V components of the Stokes vector, $\mathbf{I} = [I, Q, U, V]^T$, and *consider only* the first (radiance) component I.
- This approach is known as the **scalar approximation**, in contrast to the more accurate **vector** description.
- This scalar approximation is valid for longwave radiation where thermal emission and absorption dominate over scattering processes.
- When scattering is important, the radiation is generally partially polarized:
- An accurate description of scattering of sunlight in a clear atmosphere or in pure water requires the full Stokes vector representation.

Basic State Variables – Scalar versus vector radiance (2)

- In radiative transfer theory the **scalar radiance** plays as central a role as the **wave function** in quantum theory.
- Its full specification as a function of: (i) **position**, (ii) **direction**, and (iii) **frequency** conveys all of the desired information about the *scalar* radiation field.

A brief description of our notation:

- Radiation state variables are described in terms of both **spectral** (or **monochromatic**) quantities and frequency integrated quantities.
- Frequency, ν, is measured in cycles per second or **Hertz**, abbreviated as [Hz]. For spectral quantities, we may visualize a small frequency interval over which all properties of the radiation and its interaction with matter are constant.
- Spectral quantities can **also** be expressed as a function of wavelength, λ [nm] or [μm], or wavenumber, $\tilde{\nu} = 1/\lambda$ [cm^{-1}].

Basic State Variables – Geometrical Optics (3)

It is important to point out that:

- The basic assumptions of the radiative transfer theory are the same as those of **geometrical optics**.

A sharply defined *pencil* of radiation was first defined in geometrical optics:

- A radiation pencil is realized physically by allowing light emanating from a point source to pass through a small opening in an opaque screen. This light may be viewed by allowing it to fall on a second screen.
- If we were to examine this spot of light near its boundary, we would notice that the edge would not be geometrically sharp. Instead we would find a series of bright and dark bands, called *diffraction fringes*.
- The size of the region over which these bands occur is of the order of the wavelength of light, λ.
- If the diameter of the cross-sectional area of the pencil is very much larger than λ, **diffraction effects are small**, and we may speak of a sharply bounded pencil of rays.

Basic State Variables – Geometrical Optics (4)

- The propagation of light may then be described in purely geometrical terms, and energy transport will occur along the direction of the light rays.
- These rays are not necessarily straight lines. In general, they are curves whose directions are determined by the **gradient** of the **index of refraction**, $m = m_{\rm r} + i m_{\rm i}$.
- The real part $m_{\rm r} = c/v$ is the ratio of the speed of light in vacuum (c) to that in the medium (v). It is the most important light-matter interaction parameter in geometrical optics.
- Absorption along the ray depends upon the **imaginary** part $m_{\rm i}$ of the complex index of refraction. The fact that m varies with frequency is known as **dispersion**.
- In geometrical optics theory, **interference and diffraction of light are unimportant**. The same is true of the radiative transfer theory.
- It is usually sufficient to set the index of refraction equal to a constant value pertaining to either air or water, **ignoring both dispersion and ray bending**.

Basic State Variables – Geometrical Optics (5)

- Refraction is also important in radiative transfer through the **ocean–atmosphere interface**, at which the index of refraction changes abruptly.
- In this case, it is usually sufficient to assume that the index of refraction is unity ($m_{\rm r} \approx 1.0$) for air and equal to ($m_{\rm r} \approx 1.34$) for water.

For our purposes the concept of **incoherent** (non-interfering) **beams** of radiation is more convenient than the concept of ray pencils. We define a **beam** in analogy with a plane wave:

- **It carries energy in a specific propagation direction (the ray direction) and has infinite extent in the transverse direction.**
- We will use **ray** and **beam direction** synonymously. When a beam of sunlight is incident on a scattering medium (e.g., the Earth's atmosphere):
- it splits up into an infinite number of incoherent (non-interacting) beams propagating in different directions.

Basic State Variables – Geometrical Optics (6)

- Similarly, when a beam is incident on a diffusely reflecting surface (*e.g.*, a "rough" ocean surface or a plant canopy):
- the reflected radiation splits up in an infinite number of incoherent beams traveling in different directions.
- On the other hand, if a beam is incident on a perfectly smooth, plane interface:
- it will give rise to one reflected and one transmitted beam.
- The directions of these two beams follow from the geometrical optics laws of reflection and refraction (**Snell's Law**), while
- their states of polarization follow from **Fresnel's Equations**.
- It is also be convenient to define an **angular beam** as:
- an incoherent sum of beams propagating in various directions inside a small cone of solid angle $d\omega$ centered around the direction of propagation $\hat{\Omega}$ as illustrated in Fig. 1.

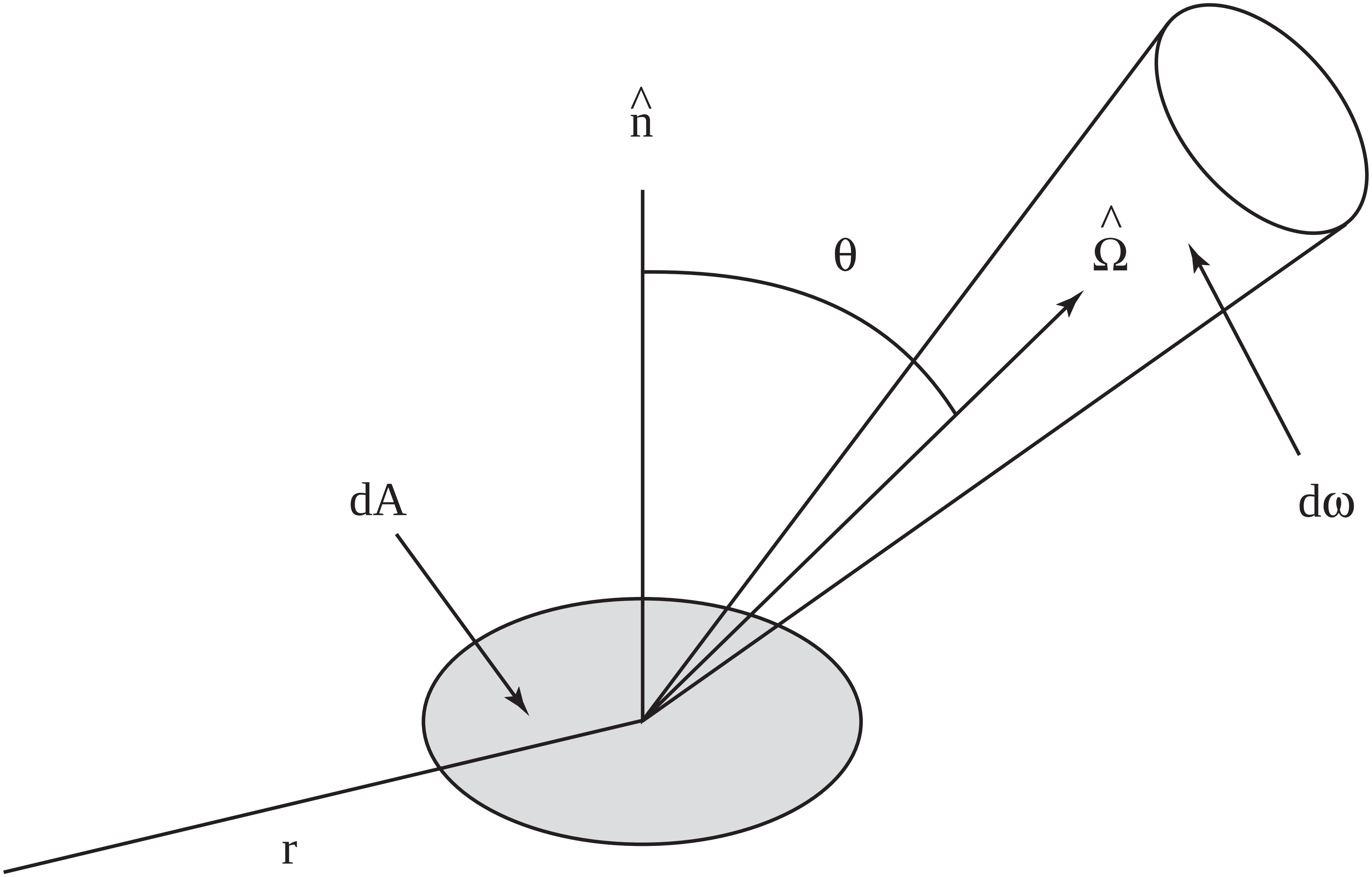


Figure 1: **The flow of radiative energy carried by a beam in the direction $\hat{\Omega}$ through a transparent surface element dA. The flow direction $\hat{\Omega}$ is at an angle θ with respect to the surface normal $\hat{n}$ ($\cos\theta = \hat{n} \cdot \hat{\Omega}$).**

Basic State Variables – Irradiance (7)

We consider the flow of radiative energy across a surface element dA, located at a specific position, and having a unit normal $\hat{n}$ (see Fig. 1).

- The energy flow is visualized as being carried by incoherent (non-interacting) beams of radiation moving in all directions.
- Because the beams traveling in different directions do not interact, we may treat them separately.
- The net rate of radiative energy flow, or power, per unit area within the small spectral range $\nu, \nu + d\nu$ is called the spectral **net irradiance**.
- We express the **spectral net irradiance** F_ν in terms of the **net** energy d^3E which crosses the area, dA, in the time interval, t, $t + dt$, within the frequency interval, $\nu, \nu + d\nu$, as:

$$F_\nu = \frac{d^3E}{dAdtd\nu} \qquad [\mathrm{W \cdot m^{-2} \cdot Hz^{-1}}].$$

Basic State Variables – Irradiance (8)

The quantities d^3E and $dAdtd\nu$ are:

- **third-order differential quantities**, considered **positive** if the flow is into the hemisphere centered on the direction $\hat{n}$, and **negative** if the flow is into the opposite hemisphere centered on $-\hat{n}$.
- A general radiation field consists of angular beams traveling in all directions.
- It is convenient to separate the energy flow into **oppositely directed, positive energy flows** in two separate hemispheres: d^3E^+ and d^3E^-.
- Each of these partial flows carries a **positive** amount of energy. We define the spectral **hemispherical irradiancees** F_ν^+ and F_ν^- as

$$F_\nu^+ = \frac{d^3E^+}{dAdtd\nu}\,; \qquad F_\nu^- = \frac{d^3E^-}{dAdtd\nu}.$$

The **net energy flow** in the positive **direction** is

$$d^3E = d^3E^+ - d^3E^-.$$

Basic State Variables – Irradiance (9)

- In the same way the spectral *net irradiance* is written as the difference of two positive quantities

$$F_\nu = F_\nu^+ - F_\nu^- .$$

Summing over all frequencies, we obtain the **net irradiance**:

$$F = \int_0^\infty d\nu F_\nu \qquad [\mathrm{W \cdot m^{-2}}].$$

- The spectral net irradiance F_λ within a small *wavelength* range λ, $\lambda + d\lambda$ is defined within the wavelength interval $d\lambda$, related to the frequency interval $d\nu$. Thus, $F_\nu\,|d\nu| = F_\lambda\,|d\lambda|$ and $(\nu = c/\lambda)$:

$$F_\lambda = F_\nu\,|d\nu/d\lambda| = F_\nu(c/\lambda^2) \qquad \left[\mathrm{W \cdot m^{-2} \cdot nm^{-1}}\right].$$

- Similarly, the spectral net irradiance per wavenumber $(\tilde{\nu} = \nu/c = 1/\lambda\ [\mathrm{cm^{-1}}])$ is:

$$F_{\tilde{\nu}} = F_\nu\,|d\nu/d\tilde{\nu}| = F_\nu c \qquad \left[\mathrm{W \cdot m^{-2} \cdot cm}\right].$$

Basic State Variables – Poynting Vector (10)

- The spectral net irradiance may also be considered to be the component of a spectral **irradiance vector** $\vec{F}_\nu(\vec{r})$ which points in the reference direction $\hat{\Omega}$, i.e.:

$$\vec{F}_\nu(\vec{r}) = F_\nu(\vec{r})\hat{\Omega}.$$

- $\vec{F}_\nu(\vec{r})$ is the generalization of the **Poynting vector** (in electromagnetic theory) to describe streams traveling in arbitrary directions.
- Clearly, the scalar spectral net irradiance is the component of $\vec{F}_\nu(\vec{r})$ in the direction $\hat{\Omega}$, that is:

$$F_\nu(\vec{r}) = \hat{\Omega} \cdot \vec{F}_\nu(\vec{r}).$$

- That part of the net irradiance originating in reflection or emission from a surface is sometimes referred to as the **radiant exitance**.

Basic State Variables – Spectral radiance and its Angular Moments (11)

- The net irradiance conveys little information about the **directional dependence** of the energy flow. A more precise description of the energy flow is desirable, especially in remote sensing applications.
- The directional dependence can be visualized in terms of a **distribution** of energy flow in all 4π directions.
- In any particular direction the energy flow is associated with the angular beam traveling in that direction.
- We denote by d^4E **any small subset of the total energy flow within a solid angle $d\omega$ around a certain direction** $\hat{\Omega}$ in the time interval dt and over a small increment of frequency $d\nu$, or energy.
- We require that this subset of radiation has passed through a surface element dA whose orientation is defined by its unit normal $\hat{n}$. The geometry is illustrated in Fig. 2.

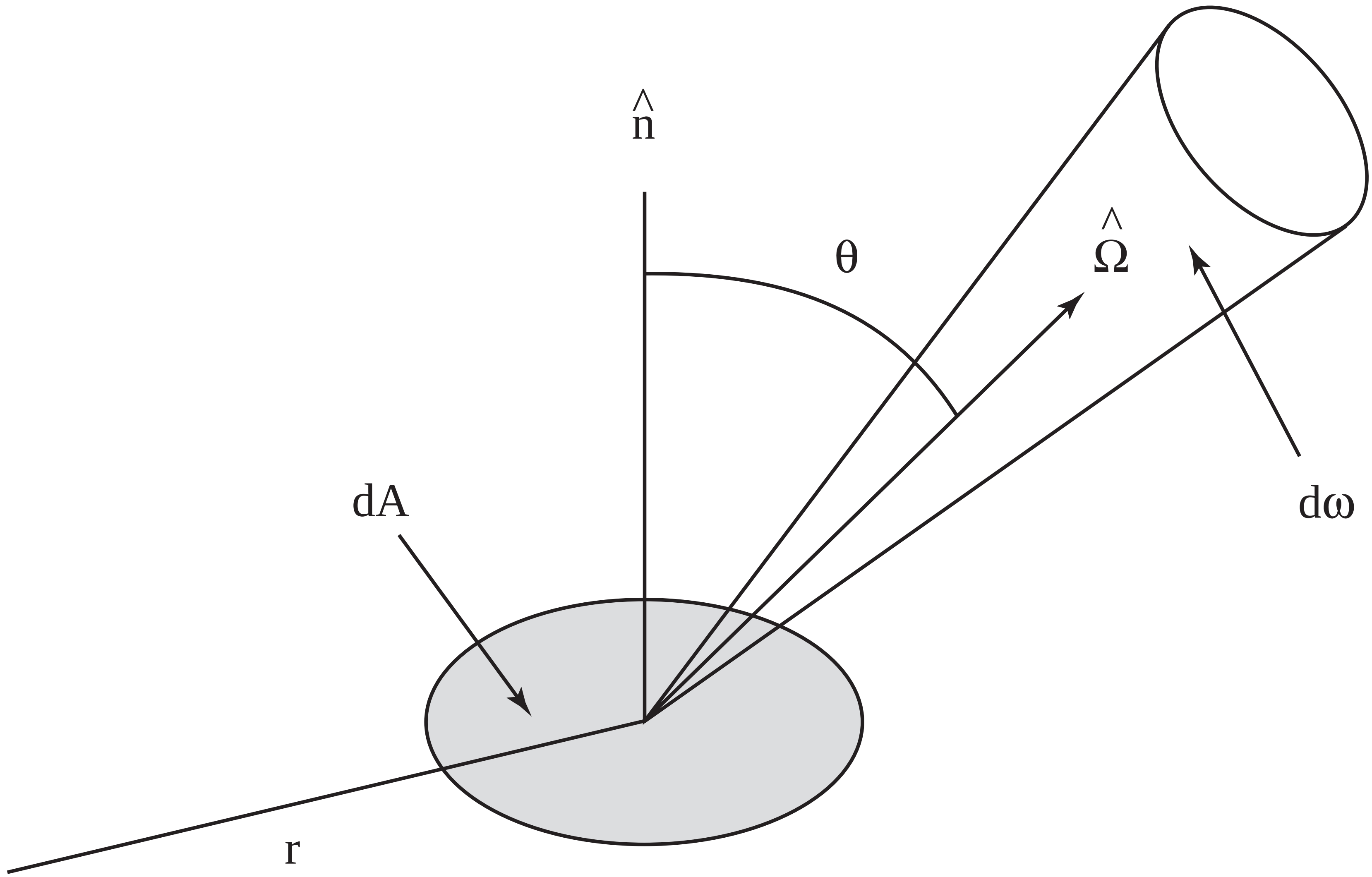


Figure 2: **The flow of radiative energy carried by a beam in the direction $\hat{\Omega}$ through a transparent surface element dA. The flow direction $\hat{\Omega}$ is at an angle θ with respect to the surface normal $\hat{n}$ ($\cos\theta = \hat{n} \cdot \hat{\Omega}$). (Same as Fig. 1.)**

Basic State Variables – Spectral radiance and its Angular Moments (12)

- The angle between $\hat{n}$ and the direction of propagation $\hat{\Omega}$ is denoted by θ. The energy per unit area, unit solid angle, unit frequency, and unit time is:

The **radiance** I_ν defined as the ratio

$$I_\nu = \frac{d^4E}{\cos\theta dA dt d\omega d\nu} \qquad \left[\mathrm{W}\cdot\mathrm{m}^{-2}\cdot\mathrm{sr}^{-1}\cdot\mathrm{Hz}^{-1}\right].$$

- Note that, in addition to dividing by $d\omega d\nu dt$, we have divided by the factor $\cos\theta = \hat{n}\cdot\hat{\Omega}$. This factor multiplied by dA is the **projection** of the surface element onto the plane normal to $\hat{\Omega}$.
- Note also that if $\hat{n}$ and $\hat{\Omega}$ are directed into opposite hemispheres, then $\cos\theta = \hat{n}\cdot\hat{\Omega}$ is negative. The energy flow is also negative in this case, by definition, so that the ratio $d^4E/\cos\theta$ remains positive.
- **Radiance is always positive.**

Basic State Variables – Spectral Radiance and its Angular Moments (13)

- Note that radiance is a **scalar quantity**, describing an angular variation of energy flow and how this angular variation itself changes with position.
- In addition to its dependence on the position variable $\vec{r}$, the angular variable $\hat{\Omega}$, and the frequency variable ν, the radiance I may also depend on time.
- Hence there are **seven** independent variables: **three** in space (x, y, z), **two** in angle (θ, ϕ), **one** in frequency (ν), and **one** in time (t).
- Except in active remote sensing applications, such as lidar and radar, **time dependence may be confidently ignored** (**seven** $\rightarrow$ **six** variables).
- Also planetary media have approximate planar uniformity. Therefore, it often **suffices to use only one position variable** (height in the atmosphere and depth in the ocean).
- Then, the number of variables is reduced from **seven** to **four**.

Basic State Variables – Relationship between Irradiance and Radiance (14)

- The relationship between Irradiance and Radiance follows from the previous equation, which we rewrite as

$$d^4E = I_\nu \cos\theta \; dA \; dt \; d\omega \; d\nu.$$

- The rate at which energy flows into each hemisphere is obtained by integration of the separate energy flows into each hemisphere, i.e.:

$$d^3E^+ = \int_+ d^4E^+; \qquad\qquad d^3E^- = \int_- d^4E^-.$$

- The subscript $(+)$ on the integral sign denotes integration over the hemisphere defined by $\hat{n}$, and $(-)$ integration over the hemisphere defined by $-\hat{n}$. The previous definitions of F_ν^+ and F_ν^- yields *positive* half-range irradiances:

$$F_\nu^+ = \frac{d^3E^+}{dAdtd\nu} = \int_+ d\omega \cos\theta I_\nu; \qquad F_\nu^- = \frac{d^3E^-}{dAdtd\nu} = -\int_- d\omega \cos\theta I_\nu.$$

Basic State Variables – Relationship between Irradiance and Radiance (15)

Combination of the half-range irradiances yields the **net irradiance:**

Integration over all solid angles yields the spectral net irradiance

$$F_\nu = F_\nu^+ - F_\nu^- = \int_{4\pi} d\omega \cos\theta I_\nu \qquad [\mathrm{W \cdot m^{-2} \cdot Hz^{-1}}].$$

- If the spectral radiance $I_\nu(\vec{r}, \hat{\Omega})$ at a point is independent of direction $\hat{\Omega}$, it is said to be **isotropic**. If it is independent of position $\vec{r}$, it is called **homogeneous**.

- The spectral radiance is both isotropic and homogeneous in the special case of **thermodynamic equilibrium**, where the **net irradiance is zero** everywhere in the medium.

- The reason is that even though the hemispherical irradiances are finite, they are of equal magnitude and opposite direction. Therefore, no **net** energy flow can occur in this equilibrium case.

Basic State Variables – Average Radiance and Energy Density (16)

- Averaging the directionally dependent radiance over all directions at a given point, $\vec{r}$, yields a scalar quantity called

The **Spectral Average (Mean) Radiance**

$$\bar{I}_\nu(\vec{r}) = (1/4\pi) \int_{4\pi} d\omega \, I_\nu(\vec{r}, \hat{\Omega}) \qquad [\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{Hz}^{-1}].$$

- The overbar ($^-$) indicates an average over the sphere. $\bar{I}_\nu$ is proportional to the spectral **energy density** of the radiation field $\mathcal{U}_\nu$, the radiative energy that resides within a unit volume.
- In the Ocean Optics community $\bar{I}_\nu$ is referred to as the "scalar irradiance".
- The 'actinic flux' used by photochemists is simply $4\pi\bar{I}_\nu$. The spectral energy density $\mathcal{U}_\nu$ can be related to the radiance in the following way.

Basic State Variables – Average Radiance and Energy Density (17)

- Consider a small cylindrical volume element of area dA having a unit normal $\hat{n}$, whose length,

$$cdt = \text{ the distance light travels in time } dt \ (c \text{ is the light speed}),$$

and let the direction of propagation $\hat{\Omega}$ be at an angle θ with respect to $\hat{n}$.

- The volume of this element is

$$dV = dA \cos\theta cdt,$$

where $dA\cos\theta$ is the projection of the surface element dA onto the plane normal to $\hat{\Omega}$.

- The energy per unit volume per unit frequency residing in the volume between t and $t + dt$ is therefore:

$$d\mathcal{U}_\nu = \frac{d^4E}{dV\,d\nu} = \frac{I_\nu \cos\theta dAdt\; d\nu\; d\omega}{dA\cos\theta cdtd\nu} = \frac{I_\nu}{c}\,d\omega.$$

Basic State Variables – Average Radiance and Energy Density (18)

- If we consider the energy density in the vicinity of a collection of incoherent beams travelling in all 4π directions, we must integrate this expression over all solid angles $d\omega$ to obtain:

$$\mathcal{U}_\nu = \int_{4\pi} d\mathcal{U}_\nu = \frac{1}{c}\int_{4\pi} d\omega\, I_\nu = \frac{4\pi}{c}\bar{I}_\nu \qquad [\mathrm{J}\cdot\mathrm{m}^{-3}\cdot\mathrm{Hz}^{-1}].$$

- The **total energy density** is the sum over all frequencies:

$$\mathcal{U} = \int_0^\infty d\nu\, \mathcal{U}_\nu \qquad [\mathrm{J}\cdot\mathrm{m}^{-3}].$$

Example 2.1 Isotropic distribution (19)

- Let us assume that the spectral radiance is independent of direction, i.e., $I_\nu(\hat{\Omega}) = I_\nu =$ constant. This assumption applies to a medium in thermodynamic equilibrium and is approximately valid deep inside a dense medium. The irradiance and energy density are easily evaluated as:

$$F_\nu^+ = F_\nu^- = \pi I_\nu$$

$$F_\nu = F_\nu^+ - F_\nu^- = 0$$

$$\mathcal{U}_\nu = \frac{4\pi I_\nu}{c}.$$

Note that:

- the two contributions to the irradiance from the opposite hemispheres cancel. Mathematically, this cancellation occurs because $\cos\theta$ is an odd function of θ in the interval $[0, \pi]$, and **physically because of a balance between upward and downward flowing beams**.

Example 2.2 Hemispherically-isotropic distribution (20)

- This example represents the simplest non-trivial description of diffuse radiation having a **non-zero** net energy transport.
- Let I_ν^+ denote the (constant) value of the radiance everywhere in the positive hemisphere, and let I_ν^- denote the (constant) value of the radiance everywhere in the opposite hemisphere. $I_\nu^+ \neq I_\nu^-$ by assumption.
- As in the isotropic case, we solve for the various moments:

$$F_\nu = I_\nu^+ \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta + I_\nu^- \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta = \pi(I_\nu^+ - I_\nu^-)$$

$$\mathcal{U}_\nu = \frac{I_\nu^+}{c} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta + \frac{I_\nu^-}{c} \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta = \frac{2\pi}{c}(I_\nu^+ + I_\nu^-).$$

- The angular distribution of radiance has been replaced by a pair of numbers at each point in the medium.
- In some situations, this approximation is too inaccurate.
- However, in situations where the spectral radiance is close to being isotropic, it leads to results of surprisingly high accuracy, as discussed in Chapter 7.

Example 2.3 Collimated Distribution (21)

- Ignoring the finite size of the Sun, we write the radiance in the direction $\hat{\Omega}$ as:

$$I_\nu^{\rm s}(\hat{\Omega}) = F_\nu^{\rm s} \delta(\hat{\Omega} - \hat{\Omega}_0) \qquad \left[\mathrm{W \cdot m^{-2} \cdot sr^{-1} \cdot Hz^{-1}}\right].$$

- $F_\nu^{\rm s}$ is the irradiance carried by the beam across a plane normal to the direction of incidence $\hat{\Omega}_0$. $\hat{\Omega}_0$ has the polar angle θ_0 and the azimuthal angle ϕ_0.
- $\delta(\hat{\Omega} - \hat{\Omega}_0) = \delta(\phi - \phi_0)\delta(\cos\theta - \cos\theta_0)$ is the two-dimensional *Dirac δ-function*. Since the delta-function has the 'units' of inverse solid angle, $I_\nu^{\rm s}$ has the correct units (energy per unit area per unit frequency per unit time per unit solid angle).

- The moments for a collimated beam are ($u \equiv \cos\theta$ and $\mu_0 \equiv \cos\theta_0$):

$$F_\nu = F_\nu^{\rm s} \int_0^{2\pi} d\phi \; \delta(\phi - \phi_0) \int_{-1}^{1} du \; u \; \delta(u - \mu_0) = F_\nu^{\rm s} \mu_0$$

$$\mathcal{U}_\nu = \frac{F_\nu^{\rm s}}{c} \int_0^{2\pi} d\phi \; \delta(\phi - \phi_0) \int_{-1}^{+1} du \; \delta(u - \mu_0) = \frac{F_\nu^{\rm s}}{c}.$$

Example 2.4 Azimuthally Symmetric Distribution (22)

- In this special case, the radiance distribution is constant as one varies the azimuthal angle ϕ, i.e., $I_\nu \neq I_\nu(\phi)$. Of course, all other variables are held constant.

- This case describes the singly scattered part of the radiation field in a coordinate system in which the z-axis is along the incoming direction of a collimated beam.

- It is also valid for the multiply scattered radiation in a **slab** or **plane-parallel** geometry when the scattering phase function is **isotropic**. Further, it applies to a radiation field produced by thermal emission, and of course, to an isotropic or hemispherically isotropic distribution.

- The angular moments are:

$$F_\nu = \int_0^{2\pi} d\phi \int_{-1}^{+1} du \; u I_\nu(u) = 2\pi \int_{-1}^{+1} du \; u I_\nu(u)$$

$$\mathcal{U}_\nu = \frac{1}{c} \int_0^{2\pi} d\phi \int_{-1}^{+1} du \; I_\nu(u) = (4\pi/c)\bar{I}_\nu.$$

Some Theorems on Radiance (23)

- Perhaps the most important property of the radiance is expressed in the following theorem (see Fig. 3):

Theorem I:

In a transparent medium, the radiance is constant along a ray.

- **Theorem I** may be generalized to apply to a beam which is reflected any number of times by perfectly-reflecting mirrors:

Theorem II:

The radiance remains constant along a ray upon perfect reflection by any mirror or combination of mirrors.

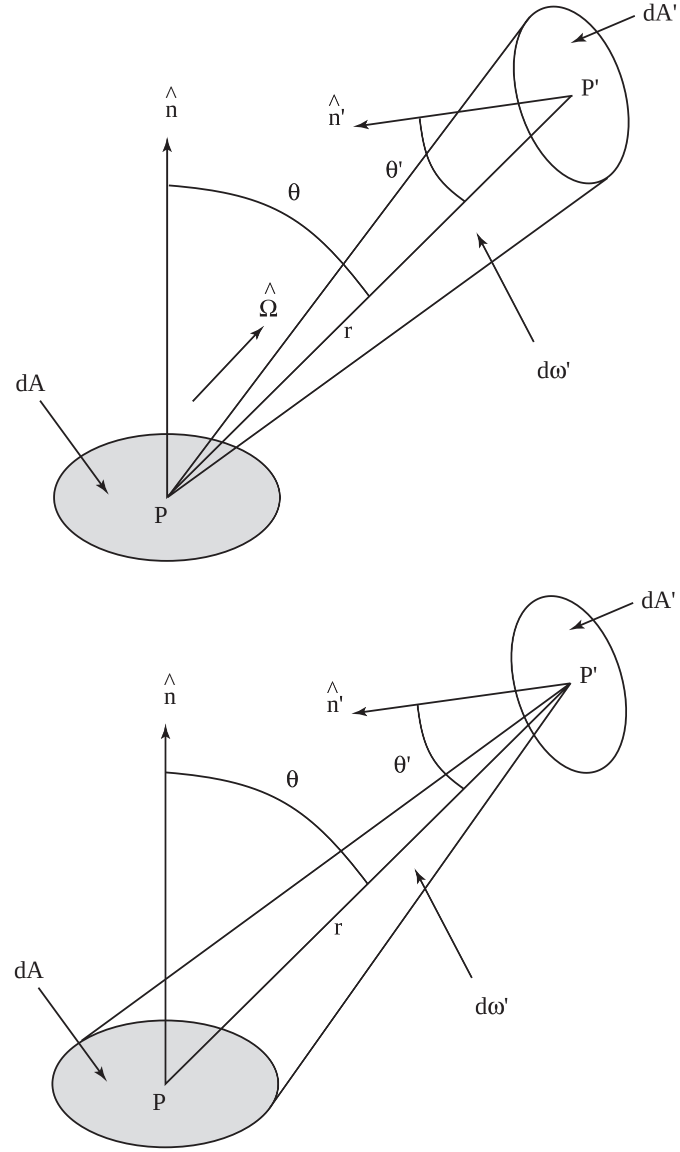


Figure 3: **The energy crossing a transparent area dA and entering into solid angle $d\omega$ is the same as that which crosses the area dA' and is contained within the solid angle $d\omega'$. r is the distance between P and P'.**

Some Theorems on Radiance (24)

- A third property of the radiance applies to refraction in a transparent medium of variable index of refraction. The theorem applies for discrete changes in $m(\nu)$ as long as reflection at the interfaces can be neglected:

Theorem III:
The quantity $I_\nu/m^2(\nu)$ remains constant along a ray in a transparent medium, provided that the reflectance at each interface can be neglected.

- The quantity $I_\nu/m^2(\nu)$ is called the *basic radiance.* Clearly, **Theorem I** is a special case of **Theorem III**.

The Extinction Law (25)

- We now introduce the specific interaction properties which are essential elements in the radiative transfer theory.
- They are defined in terms of the most important principle in the theory, the **Extinction Law**, more commonly known as **Beer's Law**, the **Beer-Lambert Law**, or the **Beer-Lambert-Bouguer Law**.
- Consider a small volume dV containing matter described by $n \quad [\mathrm{m}^{-3}]$, the **concentration**, defined as the ratio of the number of particles dN divided by the volume dV.
- The particles are assumed to be **optically significant**, that is, they have a non-negligible effect on radiation that passes through the volume.
- Other (optically insignificant) particles may be present, but they may be ignored for the present purposes.

The Extinction Law (26)

- For convenience, the volume dV is considered to be a slab of infinitesimal thickness ds and area dA (Fig. 4). Suppose a beam of radiation is incident normally on the slab. From Eq. (2.3) the differential of energy falling on the front surface is:

$$d^4E = I_\nu \, dA \, dt \, d\nu \, d\omega.$$

- As the beam of radiation passes through the slab, it interacts with the particles through either absorption or scattering and a reduced amount of energy emerges at the opposite side. The beam of radiation is said to have suffered **extinction**.

- The energy emerging from the back face of the slab is given by the original energy, less an amount given by:

$$\delta(d^4E) = dI_\nu \, dA \, dt \, d\nu \, d\omega$$

where dI_ν is the differential loss in radiance along the length ds.

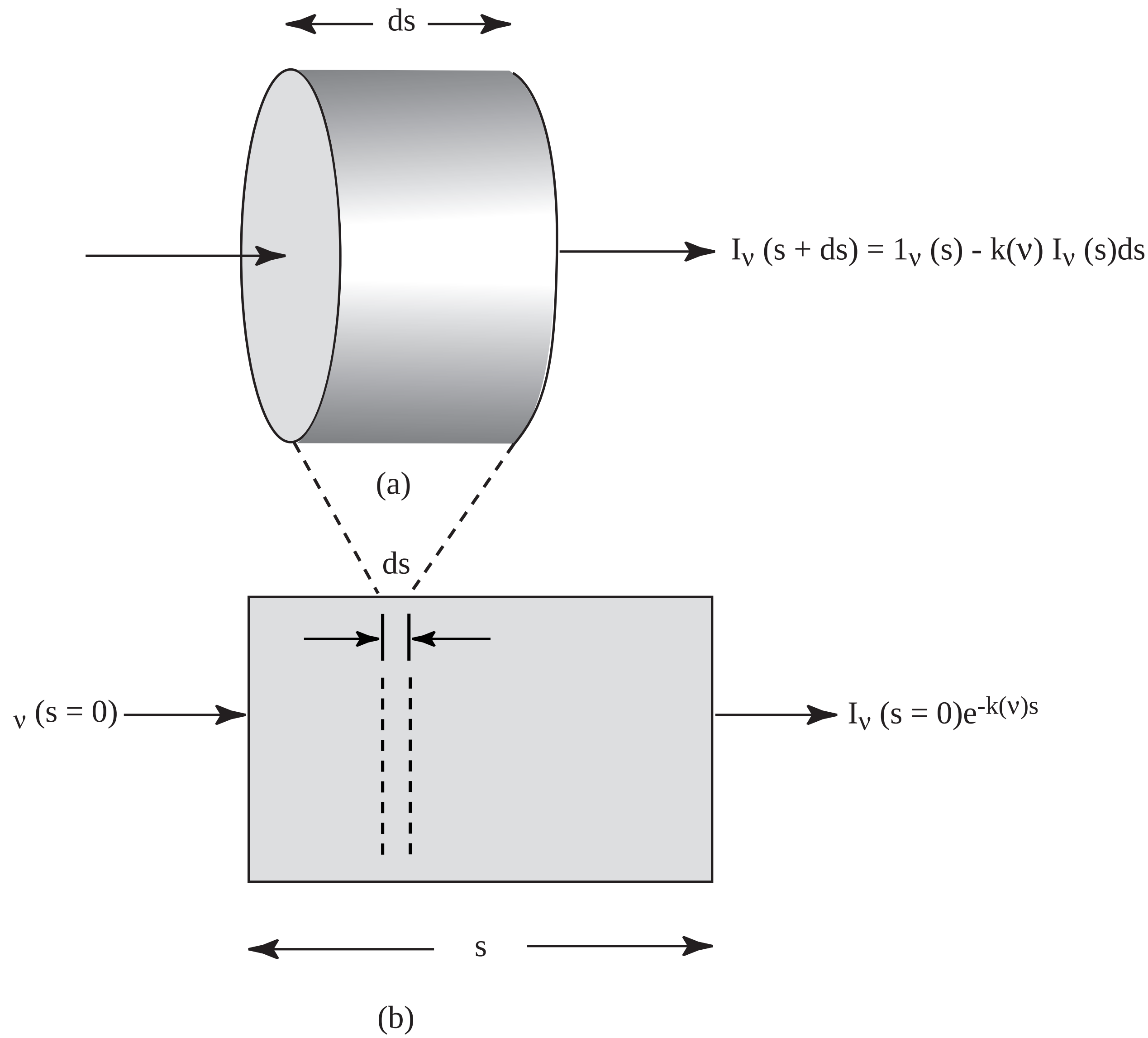


Figure 4: **(a) Radiance passing through a thin slab suffers extinction proportional to the path length ds. (b) Radiance passing through a finite path length suffers exponential extinction.**

The Extinction Law (27)

- It is found experimentally that the degree of **weakening depends linearly upon both the incident radiance** and the amount of optically active matter along the beam direction (proportional to the length ds):

The Extinction Law (Differential Form)

$$dI_\nu \propto -I_\nu ds.$$

- We define the constant of proportionality to be the **extinction coefficient** k.

The Extinction Law (28)

There are three different ways of defining extinction, in terms of:

- the path length itself ds;
- the *mass path* $d\mathcal{M} = \rho ds$; or
- the *column number* $d\mathcal{N} = nds$

of the absorbing/scattering species concentration.

- Here ρ is **mass density** $[\mathrm{kg} \cdot \mathrm{m}^{-3}]$ and
- n is **particle density** or **concentration** $[\mathrm{m}^{-3}]$ of the optically significant gas.

The Extinction Law (29)

- Hence, the following three definitions of extinction are obtained:

$$k(\nu) \equiv -\frac{dI_\nu}{I_\nu ds} \qquad \text{extinction coefficient } [\mathrm{m}^{-1}]$$

$$k_m(\nu) \equiv -\frac{dI_\nu}{I_\nu \rho ds} = -\frac{dI_\nu}{I_\nu d\mathcal{M}} \quad \text{mass extinction coefficient } [\mathrm{m}^2 \cdot \mathrm{kg}^{-1}]$$

$$k_n(\nu) \equiv -\frac{dI_\nu}{I_\nu n ds} = -\frac{dI_\nu}{I_\nu d\mathcal{N}} \qquad \text{extinction cross section } [\mathrm{m}^2].$$

Note that:

- the extinction cross section k_n is analogous to the collision cross section in atomic physics.
- It is interpreted as the **effective area** of the molecule or particle presented to the incident beam, resulting in some of the beam being absorbed or deflected (scattered) into other directions.

The Extinction Law (30)

- Let the total length of the straight-line path through an extended section of the medium be s, and an intermediate path length be $s' \leq s$.
- Denoting the radiance entering the medium at $s' = 0$ as $I_\nu(s' = 0, \hat{\Omega})$, we need to find the radiance $I_\nu(s' = s, \hat{\Omega})$, where $\hat{\Omega}$ denotes the propagation direction of the beam. Integrating the equations above from $s' = 0$ to $s' = s$, we obtain:

$$\tau_s(\nu) = -\ln\left[\frac{I_\nu(s' = s, \hat{\Omega})}{I_\nu(s' = 0, \hat{\Omega})}\right]$$

where

$$\tau_s(\nu) \equiv \int_0^s ds'\, k(\nu) \equiv \int_0^s ds'\, k_m(\nu)\rho \equiv \int_0^s ds'\, k_n(\nu)n.$$

- Here τ_s is the **extinction optical path** or *opacity* along the path s.
- τ_s (a dimensionless quantity) is a measure of the strength and number of optically significant particles along a beam.

The Extinction Law (31)

- Solving for the radiance at $s' = s$ by taking the antilogarithm we obtain:

The Extinction Law (integrated form)

$$I_\nu(s, \hat{\Omega}) = I_\nu(0, \hat{\Omega}) \exp\left[-\tau_s(\nu)\right].$$

- The radiance is seen to decay exponentially with optical path along the beam direction.
- The Extinction Law reduces to **Theorem I** when the optical path is zero, resulting in the statement that
- the radiance remains constant along the beam direction in the absence of extinction.

Extinction = Scattering plus Absorption (32)

- We have dealt with extinction as if it were one process, whereas it is actually caused by two distinctly different phenomena.
- It is clear that the attenuation of a light beam in a specific direction can be obtained by either absorption or scattering.
- This attenuation is obvious for absorption but some care needs to be given as to how scattering also weakens the beam. Since this process diverts the radiation into beams propagating in other directions, it must necessarily result in a loss of energy in the initial beam along $\hat{\Omega}$.
- However, suppose a photon in the beam is 'deflected' only a very small amount. A detector of finite angular resolution may then measure the presence of this scattered photon along with the original (unscattered) beam photons.
- This small deflection can be a difficult experimental problem, since small-angle scattering from particulate matter (the so-called "diffraction peak") can be orders of magnitude more efficient than large-angle scattering.

Extinction = Scattering plus Absorption (33)

- It is also possible that photons in a different beam, propagating in direction $\hat{\Omega}''$, might be deflected **into** the direction $\hat{\Omega}$, and thus be confused with the original beam. This deflection is a consequence of **multiple scattering**.
- The extinction optical path τ_s of a mixture of scattering/absorbing molecules and particles is defined as the sum of the individual scattering optical path, $\tau_{sc}(\nu)$, and the absorption optical path, $\tau_a(\nu)$, i.e.

$$\tau_s(\nu) = \tau_{sc}(\nu) + \tau_a(\nu)$$

where

$$\tau_{sc}(\nu) = \sum_i \int_0^s ds'\, \sigma^i(\nu, s') = \sum_i \int_0^s ds'\, \sigma_m^i(\nu)\rho_i(s') = \sum_i \int_0^s ds'\, \sigma_n^i(\nu) n_i(s')$$

$$\tau_a(\nu) = \sum_i \int_0^s ds'\, \alpha^i(\nu, s') = \sum_i \int_0^s ds'\, \alpha_m^i(\nu)\rho_i(s') = \sum_i \int_0^s ds'\, \alpha_n^i(\nu) n_i(s').$$

Extinction = Scattering plus Absorption (34)

- The sum is over all optically significant species. ρ_i and n_i are the mass densities and concentrations of the ith optically significant species (either molecule or particle).

The quantities $\alpha^i, \alpha^i_m, \alpha^i_n$ are called:

- the **absorption coefficient**, α^i,
- the **mass absorption coefficient**, α^i_m, and
- the **absorption cross section**, α^i_n,

of the ith constituent (molecule or particle), respectively.

Similarly, the quantities $\sigma^i, \sigma^i_m, \sigma^i_n$ are called:

- the **scattering coefficient**, σ^i,
- the **mass scattering coefficient**, σ^i_m, and
- the **scattering cross section**, σ^i_n.

The Differential Equation of Radiative Transfer (35)

- We first define in a formal way the emission of radiative energy by a differential volume element within the medium. We ignore any time dependence of the radiation field.

- Consider again a slab of thickness ds, and cross-sectional area dA, filled with an optically significant material **emitting** radiative energy of frequency ν in time dt.

- This energy emerges from the slab as an **angular beam** with directions within the solid angle $d\omega$ around $\hat{\Omega}$.

- The **emission coefficient** is defined as the ratio:

$$j_\nu(\vec{r}, \hat{\Omega}) = \frac{d^4 E}{dA\ ds\ dt\ d\nu\ d\omega} = \frac{d^4 E}{dV\ dt\ d\nu\ d\omega} \qquad [\mathrm{W} \cdot \mathrm{m}^{-3} \cdot \mathrm{Hz}^{-1} \cdot \mathrm{sr}^{-1}].$$

The Differential Equation of Radiative Transfer (36)

- We now have general definitions for both the **loss** and the **gain** of radiative energy of a beam, and may therefore write the **net rate of change** of the radiance along the beam direction.

- Combining the **Extinction Law** with the definition of emission, we have:

$$dI_\nu = -k(\nu) I_\nu ds + j_\nu ds \qquad [\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{Hz}^{-1} \cdot \mathrm{sr}^{-1}]$$

where $k(\nu) = \sigma(\nu) + \alpha(\nu)$.

- Dividing by $k(\nu)ds$, the differential optical path $d\tau_s$, we find:

$$\frac{dI_\nu}{d\tau_s} = -I_\nu + \frac{j_\nu}{k(\nu)}.$$

- The ratio $j_\nu / k(\nu)$ is called the **source function**:

$$S_\nu \equiv \frac{j_\nu}{k(\nu)} \qquad [\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{Hz}^{-1} \cdot \mathrm{sr}^{-1}].$$

The Differential Equation of Radiative Transfer (37)

- Our fundamental equation may be written in three mathematically equivalent ways:

The differential equation of radiative transfer:

$$\frac{dI_\nu}{d\tau_s} = -I_\nu + S_\nu; \qquad \frac{dI_\nu}{ds} = -k(\nu)I_\nu + k(\nu)S_\nu;$$

$$\hat{\Omega} \cdot \nabla I_\nu = -k(\nu)I_\nu + k(\nu)S_\nu.$$

- In the third form the gradient operator ∇ emphasizes that we are describing a **rate of change** of radiance along the beam in the direction $\hat{\Omega}$.
- $\hat{\Omega} \cdot \nabla I_\nu$ is sometimes called the *streaming term.* A derivation of the extra term $(1/c)\partial I_\nu/\partial t$ that must be added to the LHS of the above equations when time dependence cannot be ignored is provided in Example 2.6.